ON THE STRUCTURE OF SEPARABLE INFINITE DIMENSIONAL BANACH SPACES

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1. Introduction

Let X be a separable infinite dimensional real Banach space. There are three general types of questions we often ask.

1. What can be said about the structure of X itself?
2. Can one find a "nice" subspace $Y \subseteq X$?
3. Does X embed into a "nice" superspace $X \subseteq Z$?

X could be arbitrary, X might have some specific property or it might be completely defined, e.g., $X = \ell_p$ $(1 \le p < \infty)$, $X = c_0$, $X = L_p$ $(1 \le p < \infty)$, $X = C[0,1]$ or $X = C(K)$, $K =$ compact metric, the standard Banach spaces one first encounters when studying functional analysis.

Notation. $X, Y, Z, \ldots$ will always denote separable infinite dimensional real Banach spaces unless specified otherwise. X is *isomorphic* to Y $(X \sim Y)$ if there exists a one-to-one bounded linear operator $T : X \xrightarrow{\text{onto}} Y$ (hence $T^{-1} : Y \to X$ is bounded). Thus there exist $0 < c$, $C < \infty$ such that for all $x \in X$

$$c\|x\| \le \|Tx\| \le C\|x\|$$

in which case we write $X \overset{K}{\sim} Y$ where $K = Cc^{-1}$. We write $X \overset{K}{\hookrightarrow} Y$ if $X \overset{K}{\sim} Z$ for some $Z \subseteq Y$.

The *Banach-Mazur distance* between isomorphic spaces X and Y is $d(X,Y) = \inf\{K : X \overset{K}{\sim} Y\}$.

Caution. To get a metric you need to take $\log(d(X,Y))$ and identify X with all Y's satisfying $d(X,Y) = 1$. The Banach-Mazur distance satisfies $d(X,Z) \le d(X,Y)d(Y,Z)$ rather than the triangle inequality if $X \sim Y \sim Z$.

We first consider our three questions in terms of a basis, as an illustration.

We have generally chosen to cite original references. There are many excellent books containing a large number of the results we present, both basic and more advanced. Among these are [LT1, LT2], [AKa], [D], [JL1, JL2], [BL], and [FHHSPZ].

Definition. $(e_i)_{i=1}^\infty$ is a *basis* for X if for all $x \in X$ there exists a unique sequence of reals $(a_i)_{i=1}^\infty$ so that

$$x = \sum_{i=1}^\infty a_i e_i \ .$$

Proposition 1.1. *Let $(e_i)_{i=1}^\infty \subseteq X$. Then (e_i) is a basis for X iff*

i) *$e_i \neq 0$ for all i*
ii) *$[(e_i)] \equiv$ the closed linear span of $(e_i) = X$*
iii) *There exists $K < \infty$ so that for all $n < m$ in $\mathbb{N}$ and $(a_i)_1^m \subseteq \mathbb{R}$*

$$\Big\|\sum_1^n a_i e_i\Big\| \le K \Big\|\sum_1^m a_i e_i\Big\|$$

In this case the smallest such K is called the *basis constant* of (e_i) and we say (e_i) is K-basic.

(e_i) is *monotone* if $K = 1$.

$(e_i) \subseteq X$ is *basic* if (e_i) is a basis for $[(e_i)]$.

Thus if (e_i) is a basis for X the basis projections $(P_n)_{n=1}^\infty$, given by $P_n(\sum_1^\infty a_i e_i) = \sum_1^n a_i e_i$ are uniformly bounded and the basis constant of (e_i) is $\sup_n \|P_n\|$. From this we obtain that the coordinate functionals (e_i^*), given by $e_i^*(\sum a_j e_j) = a_i$, are elements of X^*. For every $x \in X$, $x = \sum_1^\infty e_i^*(x) e_i$.

Sometimes it is more convenient to use the *projection constant* of a basis (e_i) given by $\sup_{n \le m} \|P_{[n,m]}\|$, where

$$P_{[n,m]}\Big(\sum a_i e_i\Big) = \sum_{i=n}^m a_i e_i \ .$$

(e_i) is *bimonotone* if this constant is 1.

It is worth noting that if (e_i) is a basis for X then we can renorm X so that (e_i) is monotone, by setting

$$\Big|\!\Big|\!\Big|\sum a_i e_i\Big|\!\Big|\!\Big| = \sup_n \Big\|P_n\Big(\sum a_i e_i\Big)\Big\|$$

or even bimonotone by setting

$$\Big|\!\Big|\!\Big|\sum a_i e_i\Big|\!\Big|\!\Big| = \sup_{n \le m} \Big\|P_{[n,m]}\Big(\sum a_i e_i\Big)\Big\| \ .$$

By "renorming" X we mean that $|\!|\!| \cdot |\!|\!|$ is an *equivalent* norm on X, i.e., for some c, C in $(0, \infty)$ and all $x \in X$,

$$c|\!|\!|x|\!|\!| \le \|x\| \le C|\!|\!|x|\!|\!| \ .$$

Note that if (e_i) is a normalized basis for X and K is the projection constant of (e_i) then for all scalars (a_i),

$$K^{-1} \sup_i |a_i| \le \Big\|\sum a_i e_i\Big\| \le \sum_i |a_i| \ .$$

In other words the space X sits between the c_0 and ℓ_1 norms.

All of the classical spaces listed above have monotone bases. The *unit vector basis* (e_i) given by $e_i(j) = \delta_{i,j}$ is a basis for ℓ_p, $1 \le p < \infty$ and for c_0. The *Haar basis* (h_i) is a monotone basis for L_p, $1 \le p < \infty$, and the *Schauder basis* (f_i) is a monotone basis for $C[0,1]$.

Basic sequences (x_i) and (y_i) are K-*equivalent* if for some $c^{-1}C \le K$

$$c\Big\|\sum a_i y_i\Big\| \le \Big\|\sum a_i x_i\Big\| \le C\Big\|\sum a_i y_i\Big\|$$

for all scalars (a_i). We denote this by $(x_i) \overset{K}{\sim} (y_i)$. Equivalently $T : [(x_i)] \to [(y_i)]$, given by $Tx_i = y_i$ for all i, is an isomorphism with $\|T\|\,\|T^{-1}\| \le K$. We let

$$d_b\big((x_i),(y_i)\big) = \|T\|\,\|T^{-1}\|$$

define the "basis distance" between (x_i) and (y_i).

If (e_i) is a basis for X, a *block basis* of (e_i) is a nonzero sequence $(x_i)_{i=1}^\infty$ given by

$$x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j e_j$$

for some sequence of integers $0 = n_0 < n_1 < \cdots$ and scalars (a_i). (x_i) is necessarily basic with basis constant not exceeding the basis constant of (e_i).

Perturbations of basic sequences are basic sequences equivalent to the original.

Proposition 1.2. *Let (y_i) be a normalized basic sequence in X and let (x_i) satisfy*

$$\sum_{i=1}^\infty \|y_i - x_i\| \equiv \lambda < \frac{1}{2K}$$

where K is the basis constant of (y_i). Then (x_i) is basic and $(x_i) \overset{C(\lambda)}{\sim} (y_i)$ where $C(\lambda)$ approaches 1 as $\lambda \downarrow 0$.

We will have more preliminary remarks shortly but first let us examine our three general questions in terms of "nice" meaning "has a basis".

P. Enflo [E1] proved in 1973 that not every X has a basis (see [LT1], p.29). But, as known to Banach [B], every $X \overset{1}{\hookrightarrow} C[0,1]$ which has a basis. Indeed $K = (B_{X^*}, \omega^*)$, the *unit dual ball* in the ω^*-topology, is compact metric. Every compact metric space is a continuous image of the Cantor set Δ, say $f : \Delta \to K$ is a continuous surjection. Define $T : X \to C(\Delta)$ by $Tx = x|_K \circ f$ to get an into isometry of X into $C(\Delta)$. But $C(\Delta) \overset{1}{\hookrightarrow} C[0,1]$ by an easy extension argument. So question # 1 has a negative answer for "basis" and # 3 has a positive one. Every X contains a basic sequence so # 2 also has a positive answer. Indeed we may regard $X \subseteq C[0,1]$ and then it is easy to construct a sequence $(x_n) \subseteq S_X$, the *unit sphere* of X, with for all m, $x_m \in [(f_i)_{i=m}^\infty]$, where (f_i) is the Schauder basis for $C[0,1]$. Hence a subsequence (y_i) of (x_i) is a perturbation of a block basis of (f_i) and hence is basic. In fact given $\varepsilon > 0$ one can get (y_i) to be $1+\varepsilon$-basic.

It is worth noting that by similar arguments we have that

- If $X \subseteq Y$ and Y has a basis (y_i) then X contains a basic sequence (x_i) which is a perturbation of a normalized block basis of (y_i).
- If (x_i) is normalized weakly null in X then a subsequence is $1+\varepsilon$-basic and if $X \subseteq Y$ as above it is also a perturbation of a normalized block basis of (y_i).

While not every X has a basis, X always possesses a weaker structure.

Theorem 1.3 ([OP], [Pe2])**.** *For all X and $\varepsilon > 0$ there exists a biorthogonal system $(x_n, x_n^*)^\infty \subseteq X \times X^*$ with $\|x_n\| = 1$ and $\|x_n^*\| < 1 + \varepsilon$ so that $[(x_n)] = X$ and $\langle (x_n^*) \rangle \equiv$ the linear span of (x_n^*) is ω^*-dense in X^*.*

So we can always find some sort of weak coordinate system.

In general not much can be said in regard to # 1 for an arbitrary X and not much more can be said about # 3. We shall have to specify more structure to get results in these cases. # 2 was the source of much research which we will discuss later.

A basis (e_i) is K-*unconditional* if for all scalars (a_i) and all $\varepsilon_i = \pm 1$

$$\Big\| \sum \varepsilon_i a_i e_i \Big\| \le K \Big\| \sum a_i e_i \Big\| \ .$$

The smallest such K is the *unconditional basis constant* of (e_i). This is equivalent to the existence of C so that for all scalars (a_i) and $F \subseteq \mathbb{N}$

$$\Big\| \sum_F a_i e_i \Big\| \le C \Big\| \sum a_i e_i \Big\| \ .$$

The smallest such C is the *suppression unconditional constant* of (e_i) and (e_i) is called C-suppression unconditional. Easily, if (e_i) is K-unconditional it is K-suppression unconditional and if it is C-suppression unconditional it is $2C$-unconditional. One can show

Proposition 1.4. *Let (e_i) be basic. Then (e_i) is unconditional iff whenever $\sum a_i e_i = x$ and π is any permutation of $\mathbb{N}$ then $\sum a_{\pi(i)} e_{\pi(i)} = x$.*

In other words the convergence of $\sum a_i e_i$ to X is "unconditional".

Clearly the unit vector basis (e_i) is a 1-unconditional basis for ℓ_p $(1 \le p < \infty)$ and for c_0. The Haar basis (h_i) is an unconditional basis for L_p iff $1 < p < \infty$ (a more difficult result, see [Bu1]). The results above on block bases and perturbations pass to unconditional bases.

Proposition 1.5. *Let X have an unconditional basis. Then X contains a subspace Y isomorphic to c_0 or ℓ_1 or else X is reflexive.*

The proof involves two important notions.

Definition. Let (e_i) be a basis for X.

a) (e_i) is *boundedly complete* if whenever $(a_i) \subseteq \mathbb{R}$ satisfies $\sup_n \| \sum_1^n a_i e_i \| < \infty$ then $\sum_1^\infty a_i e_i$ converges.

b) (e_i) is *shrinking* if every normalized block basis (x_i) of (e_i) is weakly null (i.e., $x^*(x_i) \to 0$ for all $x^* \in X^*$).

The unit vector basis (e_i) for ℓ_p $(1 < p < \infty)$ is both shrinking and boundedly complete. In ℓ_1 it is boundedly complete but not shrinking and in c_0 it is shrinking but not boundedly complete.

If (e_i) is a basis for X then the biorthogonal (or coordinate functionals) (e_i^*) are a basic sequence in X^* with basis constant not exceeding that of (e_i). They also form a ω^*-basis for X^*; for all $x^* \in X^*$ there exist unique scalars (a_i) with $x^* = \omega^* - \lim_{n\to\infty} \sum_{i=1}^n a_i e_i^*$. Of course, $a_i = x^*(e_i)$. One can show that a basis (e_i) for X is shrinking iff $[(e_i^*)] = X^*$, and so (e_i^*) is actually a basis for X^*.

It is also not hard to show that if (e_i) is a boundedly complete basis for X then X is isomorphic to a dual space, namely Y^* where $Y = [(e_i^*)] \subseteq X^*$.

If (e_i) is a shrinking basis for X then (e_i^*) is a boundedly complete basis for X^*. If (e_i) is a boundedly complete basis for X then (e_i^*) is a shrinking basis for $[(e_i^*)]$.

X is reflexive iff B_X is weakly compact iff B_X is weakly sequentially compact (i.e., for all bounded $(x_i) \subseteq X$ there is a subsequence (y_i) and $y \in X$ with $y_i \xrightarrow{\omega} y$, i.e., $x^*(y_i) \to x^*(y)$ for all $x^* \in X^*$). To prove Proposition 1.4 we have James' result [J1].

Proposition 1.6. *Let X have a basis (e_i). X is reflexive iff (e_i) is boundedly complete and shrinking.*

Indeed if (e_i) is boundedly complete and shrinking and $(x_i) \subseteq B_X$ then passing to a subsequence we may assume $\lim_i e_j^*(x_i) \equiv a_j$ exists for all j and moreover $\sup_n \| \sum_1^n a_j e_j \| < \infty$ so $x \equiv \sum_1^\infty a_j e_j \in X$. It follows from (e_i) is shrinking that $x_i - x \xrightarrow{\omega} 0$ so $x_i \xrightarrow{\omega} x$. The converse is also easy.

Proof of Proposition 1.4. Let (e_i) be an unconditional basis for X. If (e_i) is not boundedly complete we obtain a seminormalized block basis (x_i), i.e., $0 < \inf \|x_i\| \le \sup \|x_i\| < \infty$, of (e_i) with

$$\sup_{\varepsilon_i = \pm 1} \Big\| \sum_1^n \varepsilon_i x_i \Big\| \equiv K < \infty$$

It follows easily that for all $x^* \in B_{[(x_i)]^*}$, $\sum_1^\infty |x^*(x_i)| \le K$. Thus if $\|x^*\| = 1$ and x^* norms $\sum a_i x_i$ we have, $\| \sum_1^\infty a_i x_i \| = x^*(\sum_1^\infty a_i x_i) \le K \sup |a_i|$, so we see that (x_i) is equivalent to the unit vector basis of c_0.

If (e_i) is not shrinking we obtain a normalized block basis (x_i) and $x^* \in B_{X^*}$ with $x^*(x_i) > \lambda > 0$ for all i. Since (x_i) is unconditional, say with unconditional constant C,

$$\begin{aligned} \lambda \sum |a_i| &\le x^* \Big(\sum (\text{sign } a_i) a_i x_i \Big) \\ &\le C \Big\| \sum a_i x_i \Big\| \le C \sum |a_i| \end{aligned}$$

and so (x_i) is equivalent to the unit vector basis of ℓ_1.

In terms of "nice" meaning our space has an unconditional basis, our questions # 2 and # 3 become

\# 2. Does every X contain an unconditional basic sequence?
(No – [Gowers, Maurey 1993]; more about this later.)

\# 3. Does every X embed into a Y with an unconditional basis?

This is easily shown to be false in a number of ways. We will show it in an unorthodox manner for future purposes.

If (x_i) is a normalized weakly null sequence in a space with an unconditional basis then a subsequence of (x_i) is unconditional by our earlier remarks. We shall show this is false, in general, and thus deduce that $C[0,1]$ does not embed into a space with an unconditional basis.

The proof will also be an illustration of how to construct new Banach spaces.

Example 1.7. [MR] There exists a normalized weakly null sequence with no unconditional subsequence.

We shall define a norm on c_{00}, the linear space of all finitely supported sequences of scalars, take X to be the completion, and the weakly null sequence will be the unit vector basis (e_i) which will be a basis for X. (e_i) will have the property that the summing basis (s_i) is equivalent to a block basis of each subsequence (e_{n_i}). The *summing basis* is a basis for an isomorph of c_0 given by

$$\Big\| \sum a_i s_i \Big\| = \sup_n \Big| \sum_1^n a_i \Big| \ .$$

Since $\| \sum_1^n s_i \| = n$ but $\| \sum_1^n (-1)^i s_i \| = 1$ it is *conditional*.

The Maurey-Rosenthal example also illustrates the use of a coding in constructing a Banach space. A general procedure for constructing a norm on c_{00} so that in the completion (e_i) is a monotone normalized basis is to choose a certain set of functionals $\mathcal{F} \subseteq [-1,1]^{\mathbb{N}}$ with $e_i \in \mathcal{F}$ for all i and if $f \in \mathcal{F}$ and $n \in \mathbb{N}$ then $P_n f = f|_{[1,n]} \in \mathcal{F}$. The norm is then given for $x \in c_{00}$ by

$$\|x\|_{\mathcal{F}} \equiv \sup_{f\in\mathcal{F}} |\langle f, x\rangle| = \sup_{f\in\mathcal{F}} \Big| \sum_1^\infty f(i)x(i) \Big|$$

If $X = \overline{(c_{00}, \|\cdot\|_{\mathcal{F}})}$ then (e_n) is weakly null if $f(e_n) \to 0$ for all $f \in \overline{\mathcal{F}}$ (pointwise closure of $\mathcal{F}$) since naturally $X \subseteq C(\overline{\mathcal{F}})$. $\overline{\mathcal{F}}$ is actually the ω^*-closure of $\mathcal{F}$ in B_{X^*}.

Returning to the M-R example we let $\varepsilon_i \downarrow 0$ rapidly and choose integers $1 = m_1 < m_2 < \cdots$ so that if $i < j$ then $\frac{\sqrt{m_i}}{\sqrt{m_j}} < \varepsilon_j$. For sets of integers $E, F \subseteq \mathbb{N}$ we write $E < F$ if $\max E < \min F$. Set

$$\vec{F} = \Big\{ (F_i)_1^n : n \in \mathbb{N},\ F_i \subseteq \mathbb{N},\ F_1 < F_2 < \cdots < F_n \Big\}$$

and let $\phi : \vec{F} \to (m_i)_1^\infty$ be 1–1 (ϕ is called a coding).

Set (for $E \subseteq \mathbb{N}$, $|E|$ denotes the cardinality of E)

$$\mathcal{F} = \bigg\{ f = \sum_{i=1}^n \frac{1_{E_i}}{\sqrt{|E_i|}} : n \in \mathbb{N},\ \ (E_i)_1^n \in \vec{F},\ \ |E_1| = 1 \\ \text{and } \ |E_{i+1}| = \phi(E_1, \ldots, E_i) \bigg\} .$$

Note that if $|E| = m_i$, $|F| = m_j$ with $i < j$ then

$$\Big\langle \frac{1_E}{\sqrt{|E|}}, \frac{1_F}{\sqrt{|F|}} \Big\rangle \le \frac{m_i}{\sqrt{m_i}\sqrt{m_j}} = \frac{\sqrt{m_i}}{\sqrt{m_j}} < \varepsilon_j \ .$$

Then $X_{\mathrm{MR}} = \overline{(c_{00}, \|\cdot\|_{\mathcal{F}})}$.

From this setup it is not hard to show that given a subsequence N of $\mathbb{N}$ if $E_1 < E_2 < \cdots$ are subsets of N with $|E_1| = 1$ and $|E_{i+1}| = \phi(E_1, \ldots, E_i)$ then $x_i = \frac{1_{E_i}}{\sqrt{|E_i|}}$ is equivalent to (s_i). For all m, $(a_i)_1^m \subseteq \mathbb{R}$,

$$\sup_{n\le m} \Big| \sum_1^n a_i \Big| \le \Big\| \sum_1^m a_i x_i \Big\| \le 4 \sup_{n\le m} \Big| \sum_1^n a_i \Big| \ . \tag{1.1}$$

The left hand inequality comes from taking
$f_n = \sum_1^n \frac{1_{E_i}}{\sqrt{|E_i|}} \in \mathcal{F}$ for $n \le m$.

For the right hand inequality we use that if $f \in \mathcal{F}$ then f agrees with f_n, as defined above, for some n, then the next term could act partially on $a_{n+1}x_{n+1}$ but the ensuing terms have different associated m_j's and so are nearly orthogonal to $\sum_{n+1}^m a_i x_i$. More precisely let

$$f = \sum_1^k \frac{1_{F_i}}{\sqrt{|F_i|}} \in \mathcal{F} \quad \text{and} \quad i_0 = \max\{i : F_i = E_i\}$$

$$
\begin{aligned}
&\left|\left\langle f, \sum_1^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}}\right\rangle\right| \\
&\qquad \le \Big|\sum_1^{i_0} a_i\Big| + \left|\left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_1^m a_i \frac{1_{E_i}}{\sqrt{E_i}}\right\rangle\right| \\
&\qquad\quad + \sum_{j=i_0+1}^k \sum_{i=1}^m |a_i| \left\langle \frac{1_{E_j}}{\sqrt{|F_j|}}, \frac{1_{E_i}}{\sqrt{|E_i|}}\right\rangle .
\end{aligned}
$$

If $|F_{i_0+1}| = m_s$ the second term is

$$\le \max_i |a_i| \Big(1 + s\varepsilon_s + \sum_{s+1}^\infty \varepsilon_i\Big)$$

and the third term is

$$\le \max_i |a_i| \Big(\sum_j \Big(j\varepsilon_j + \sum_{t=j+1}^\infty \varepsilon_t\Big)\Big)$$

Equation (1.1) follows if ε_i's are sufficiently small.

We should also mention a recent result of Johnson, Maurey and Schechtman [JMS]. This is much more difficult.

Theorem 1.8. *There exists a normalized weakly null sequence (f_i) in L_1 so that for each subsequence (f_{n_i}) and $\varepsilon > 0$ the Haar basis for L_1 is $1+\varepsilon$-equivalent to a block basis of (f_{n_i}).*

Since the Haar basis for L_1 is conditional this proves that no subsequence of (f_i) is unconditional.

We will focus for a while on the problem of finding a "nice" subspace of a given X. For our future considerations of "nice" by passing to a basic sequence in X we may assume X has a basis, (e_i). In fact we could think of X as $\overline{(c_{00}, \|\cdot\|_{B_{X^*}})}$.

We need to search among all subspaces of X to find a "nice" one. The easiest search is among all subspaces $[(e_{n_i})]$ and this leads us naturally to Ramsey theory. As we shall see this will prove productive but ultimately we will have to search among all block bases for, say, an unconditional basic sequence. The Ramsey theorem that would yield a positive result fails however, of course. But partial results ensue.

2. Applications of Ramsey Theory

In Ramsey theory one generally has a structure which is finitely colored and one seeks a substructure (of sufficient size) which is monochromatic (1-colored). An example is the pigeon hole principle: if we finitely color $\mathbb{N}$ some subsequence is 1-colored. We begin with Ramsey's original result.

> For $k \in \mathbb{N}$, $[\mathbb{N}]^k$ is the set of all subsets of $\mathbb{N}$ of size k. We prefer to think of these as subsequences of length k, $(n_i)_1^k$, $n_1 < \cdots < n_k$. $[\mathbb{N}]^\omega$ denotes all (infinite) subsequences of $\mathbb{N}$. Similar definitions apply to $[M]^k$ and $[M]^\omega$ if $M \in [\mathbb{N}]^\omega$.

Theorem 2.1. [R] *Let $k \in \mathbb{N}$ and let $[\mathbb{N}]^k$ be finitely colored. Then there exists $M \in [\mathbb{N}]^\omega$ so that $[M]^k$ is 1-colored.*

Proof. We give the proof for $k = 2$ and two colors R and B. From this the general result for $k = 2$ follows and by inductive arguments, similar to the special case proof, one obtains the theorem.

Let $m_1 = 1$ and choose $M_1 \in [\mathbb{N}]^\omega$ with $1 < M_1$ (i.e., $1 < \min M_1$) so that either

a) $(m_1, m) \in R$ for all $m \in M_1$

or

b) $(m_1, m) \in B$ for all $m \in M_1$.

Let $m_2 = \min M_1$ and choose $M_2 \in [M_1]^\omega$ with $m_2 < M_2$ so that either

a) $(m_2, m) \in R$ for all $n \in M_2$

or

b) $(m_2, m) \in B$ for all $n \in M_2$.

Continue in this way and observe that either a) or b) occurs infinitely often. Choose $M \in [(m_i)_1^\infty]^\omega$ so that always a) holds, in which case $[M]^2 \subseteq R$ or always b) holds, in which case $[M]^2 \subseteq B$. □

Now this theorem deals with finite subsequences of $\mathbb{N}$ but it will still have application for us. But first let's discuss the infinite subsequence version.

Definition. $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ is *Ramsey* if for all $M \in [\mathbb{N}]^\omega$ there exists $L \in [M]^\omega$ with either $[L]^\omega \subseteq \mathcal{A}$ or $[L]^\omega \cap \mathcal{A} = \emptyset$, i.e., $[L]^\omega \subseteq \sim \mathcal{A}$.

Unlike Theorem 2.1 not every coloring of $[\mathbb{N}]^\omega$ via $\mathcal{A}$ and $\sim \mathcal{A}$ is Ramsey. Indeed Erdös and Rado gave an example as follows. Let $(M_\alpha)_{\alpha<c}$ be a well ordering of $[\mathbb{N}]^\omega$. For each $\alpha < c$ let M_α^1 and M_α^2 be distinct members of $[M_\alpha]^\omega$ which are also distinct from $\{M_\beta^1, M_\beta^2 : \beta < \alpha\}$. Let $\mathcal{A} = (M_\alpha^1)_{\alpha<c}$. So $(M_\alpha^2)_{\alpha<c} \subseteq \sim \mathcal{A}$.

The sets $\mathcal{A}$ which are Ramsey can be described topologically. Let τ be the product topology of $2^\mathbb{N}$ restricted to $[\mathbb{N}]^\omega$. Thus a base for the open sets is given by

$$\{\mathbb{N}_A : A \subseteq \mathbb{N} \text{ is finite}\} \quad \text{where}$$

$$\mathbb{N}_A \equiv \{L \in [\mathbb{N}]^\omega : A \text{ is an initial segment of } L\}.$$

Theorem 2.2. [N-W] *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be τ-open. Then $\mathcal{A}$ is Ramsey.*

The proof played a role in Gowers' proof of his dichotomy theorem (more about that later).

Proof. Given $A \in [\mathbb{N}]^{<\omega}$, i.e., A is a finite subset of $\mathbb{N}$, and $M \in [\mathbb{N}]^\omega$ we say M *accepts* A if

$$M_A \equiv \{L \in [\mathbb{N}]^\omega : L \setminus A \in [M]^\omega \text{ and}$$

$$A \text{ is an initial segment of } L\} \subseteq \mathcal{A}.$$

M *rejects* A if for all $L \in [M]^\omega$, L does not accept A.

Claim. *For all $M \in [\mathbb{N}]^\omega$ there exists $L \in [M]^\omega$ so that either*

i) *L accepts all of its finite subsets*

or

ii) *L rejects all of its finite subsets.*

Indeed we may assume no $L \in [M]^\omega$ accepts ϕ (or else i) holds). There must exist $m_1 \in M$ and $M_1 \in [M]^\omega$ so that M_1 rejects $\{m_1\}$. For if not, for every $m \in M$ and $M_0 \in [M]^\omega$ there exists $M_1 \in [M_0]^\omega$ so that M_1 accepts $\{m\}$. Inductively we can construct $(m_i) \in [M]^\omega$ so that for all k, $(m_i)_{k+1}^\infty$ accepts $\{m_k\}$. Thus $(m_i)_1^\infty$ accepts ϕ, a contradiction.

Inductively again, for all k assume we have $m_1 < \cdots < m_k$ and $M_k \in [M]^\omega$ so that M_k rejects all finite subsets of $\{m_1, \dots, m_k\}$. Then we obtain $m_{k+1} \in M_k$ and $M_{k+1} \in [M_k]^\omega$ which rejects all finite subsets of $\{m_1, \dots, m_{k+1}\}$. For if not, we obtain $(m_i)_1^\infty$ so that for all $\ell > k$, $(m_i)_{\ell+1}^\infty$ accepts $A_\ell \cup \{m_\ell\}$ for some $A_\ell \subseteq \{m_1, \dots, m_k\}$. Passing to a subsequence we may assume $A_\ell = A$ is fixed and thus $(m_i)_{k+1}^\infty$ accepts A, a contradiction. The claim is proved.

Now if i) then $[L]^\omega \subseteq \mathcal{A}$. If ii) then $[L]^\omega \cap \mathcal{A} = \emptyset$. Indeed $\mathcal{A}$ is τ-open and if not some $N_A \subseteq \mathcal{A}$, $N \in [L]^\omega$. □

Remarks. Since $\mathcal{A}$ is Ramsey iff $\sim \mathcal{A}$ is Ramsey we obtain τ-closed sets are also Ramsey. In fact Borel sets are Ramsey [GP] and analytic sets are Ramsey [Si] and even more. The proof above actually shows that $\mathcal{A}$ is Ramsey if it is open in the topology τ^*, that topology generated by the open base $\{M_A : A \text{ is finite}, M \in [\mathbb{N}]^\omega\}$. In fact one can show [El] $\mathcal{A}$ is Ramsey iff $\mathcal{A}$ has the Baire property ($\mathcal{A}$ is the symmetric difference of an open set and a meager set) for τ^*.

We turn now to applications of these theorems. The first is the notion of a spreading model due to Brunel and Sucheston [BS].

Theorem 2.3. *Let (x_n) be a normalized basic sequence in X and let $\varepsilon_n \downarrow 0$. Then there exists a subsequence (y_n) of (x_n) so that for all n, $(a_i)_1^n \subseteq [-1,1]$ and integers $n \le k_1 < \cdots < k_n$, $n \le i_1 < \cdots < i_n$,*

$$\left| \Big\| \sum_{j=1}^n a_j y_{k_j} \Big\| - \Big\| \sum_{j=1}^n a_j y_{i_j} \Big\| \right| < \varepsilon_n \ .$$

If we assume the theorem for the moment we have that for all n, $(a_i)_1^n \subseteq \mathbb{R}$,

$$\lim_{i_1 \to \infty} \dots \lim_{i_n \to \infty} \Big\| \sum_{j=1}^n a_j y_{i_j} \Big\| \text{ exists.}$$

Moreover if we denote this limit by $\| \sum_{j=1}^n a_j e_j \|$ this yields a norm on c_{00} so that $\overline{(c_{00}, \| \cdot \|)}$ is a Banach space with normalized basis (e_i). This space or (e_i) is called a *spreading model* of X or of (y_i).

Proof of Theorem 2.3. By a diagonal argument it suffices to show that for all n there exists $(y_i) \subseteq (x_i)$ so that for all $(a_i)_{i=1}^n \subseteq [-1,1]$, $k_1 < \cdots < k_n$, $i_1 < \cdots < i_n$, that

$$\Big| \Big\| \sum_{j=1}^n a_j y_{k_j} \Big\| - \Big\| \sum_{j=1}^n a_j y_{i_j} \Big\| \Big| < \varepsilon_n \ .$$

This is easily achieved for a fixed $(a_i)_{i=1}^n$ via Theorem 2.1. We partition $[0,n]$ into subintervals $(I_j)_{j=1}^m$ of length $< \varepsilon_n$ and "color" $(k_1, \ldots, k_n)$ by I_ℓ if

$$\Big\| \sum_{j=1}^n a_j y_{k_j} \Big\| \in I_\ell \ .$$

For the general case we choose a finite $\varepsilon_n/4$-net for $[-1,1]^n$ under the ℓ_1^n-metric and successively apply the above for each element of this net, and then use the triangle inequality for general $(a_i)_1^n$. □

Notation. It is convenient to denote the spreading model of (y_n) by $(\tilde y_n)$ rather than (e_n).

Proposition 2.4. *Let $(\tilde y_n)$ be a spreading model of (y_n)*

a) *$(\tilde y_n)$ is* 1-spreading*, i.e., for all n, $(a_i)_1^n \subseteq \mathbb{R}$ and integers $k_1 < \cdots < k_n$,*

$$\Big\| \sum_1^n a_i \tilde y_i \Big\| = \Big\| \sum_1^n a_i \tilde y_{k_i} \Big\| \ .$$

b) *If (y_n) is weakly null, then $(\tilde y_n)$ is suppression 1 unconditional, i.e.,*

$$\Big\| \sum_F a_i \tilde y_i \Big\| \le \Big\| \sum a_i \tilde y_i \Big\|$$

for all $F \in \mathbb{N}$, and all $(a_i) \subseteq \mathbb{R}$. Thus $(\tilde y_n)$ is subsymmetric, i.e., spreading and unconditional.

Proof. a) is obvious.

b) It suffices to prove that for $n \in \mathbb{N}$, $(a_i)_1^n \subseteq [-1,1]$ and $1 \le i_0 \le n$,

$$\Big\| \sum_{\substack{i=1 \\ i \ne i_0}}^n a_i \tilde y_i \Big\| \le \Big\| \sum_{i=1}^n a_i \tilde y_i \Big\| \ .$$

To achieve this we give an easy but useful lemma which we shall use again later.

Lemma 2.5. *Let (y_n) be normalized weakly null in X. Then for all $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ there exists $m_0 > n_0$, $m_0 = m(\varepsilon, n_0)$, so that, if $x^* \in B_{X^*}$ then there exists $n \in (n_0, m_0)$ with*

$$|x^*(y_n)| < \varepsilon \ .$$

Proof. Assume not and for all $m > n_0$ find $x_m^* \in B_{X^*}$ with $|x_m^*(y_n)| \ge \varepsilon$ for all $n \in (n_0, m)$. Let x^* be a ω^*-limit point of $(x_m^*)_{m=1}^\infty$. Then $|x^*(y_n)| \ge \varepsilon$ for all $n > n_0$ which contradicts $y_n \xrightarrow{\omega} 0$. □

So again fix $(a_i)_1^n \subseteq [-1,1]$ and i_0. Let $p \in \mathbb{N}$, $p \ge n$. Let $p \le k_1 < \cdots < k_{i_0} < \cdots < k_n$. Then

$$\Big| \Big\| \sum_1^n a_i y_{k_i} \Big\| - \Big\| \sum_1^n a_i \tilde y_i \Big\| \Big| \le \varepsilon_p \ .$$

We can choose the k_i's so that

$$k_{i_0+1} = m(\varepsilon_p, k_{i_0}) \qquad \text{(Lemma 2.5)}$$

and let $x^* \in S_{X^*}$ with

$$x^*\Big(\sum_{\substack{i=1\\ i\neq i_0}}^n a_i y_{k_i}\Big) = \Big\|\sum_{\substack{i=1\\ i\neq i_0}}^n a_i y_{k_i}\Big\| \ .$$

Choose $\bar{k} \in (k_{i_0}, k_{i_0+1})$ with $|x^*(y_{\bar{k}})| < \varepsilon_p$. Then

$$\Big\|\sum_{\substack{i=1\\ i\neq i_0}}^n a_i y_{k_i}\Big\| \overset{\varepsilon_p}{\approx} x^*\Big(\sum_{\substack{i=1\\ i\neq i_0}}^n a_i y_{k_i} + a_{i_0} y_{\bar{k}}\Big)$$

$$\leq \Big\|\sum_{\substack{i=1\\ i\neq i_0}}^n a_i y_{k_i} + a_{i_0} y_{\bar{k}}\Big\| \overset{\varepsilon_p}{\approx} \Big\|\sum_1^n a_i \tilde{y}_i\Big\|$$

where here $\overset{\varepsilon_p}{\approx}$ means the two terms differ by at most ε_p. Letting $p \to \infty$ yields the result. □

Remarks. Thus if we were searching for an unconditional basic sequence inside X and X admitted a normalized weakly null sequence (x_n) then by Proposition 2.4 we could find an unconditional spreading model $(\tilde{y}_n)$ of some $(y_n) \subseteq (x_n)$. (y_n) would be *asymptotically unconditional*: every $(y_{k_i})_1^n$ with $n \leq y_{k_1} < \cdots < y_{k_n}$ would be suppression $1+\varepsilon_n$-unconditional. So when does X admit such a sequence? It does not if $X = \ell_1$: if (x_n) were such a sequence then a subsequence would be equivalent to a normalized block basis (b_n) of the unit vector basis of ℓ_1 but (b_n) is 1-equivalent to the unit vector basis of ℓ_1 which is not weakly null. More generally, ℓ_1 is a *Schur* space: $x_n \xrightarrow{\omega} x$ iff $\|x_n - x\| \to 0$.

One of the most famous and beautiful theorems in Banach space theory yields as a corollary that if X does not contain (an isomorph of) ℓ_1 then X contains a normalized weakly null sequence.

Theorem 2.6 (Rosenthal [Ro1])**.** *Let $(x_n) \subseteq S_X$ admit no weak Cauchy subsequence. Then there exists $(y_n) \subseteq (x_n)$ which is equivalent to the unit vector basis of ℓ_1.*

Rosenthal's original proof did not use Ramsey theory although the ideas involved are closely related. Farahat [F] showed how to involve Ramsey theory in the argument.

We can regard $X \subseteq C(K)$ where $K = (B_{X^*}, \omega^*)$, a compact metric space. (x_n) has no weak Cauchy subsequence means that no subsequence is pointwise convergent on K. We regard $x_n : K \to [-1,1]$ (the continuity is not important, only the "sup" norm). The object is to find $r \in \mathbb{R}$ and $\delta > 0$ and $(y_n) \subseteq (x_n)$ so that $(A_n = [y_n > r+\delta]$, $B_n = [y_n < r])$ are *Boolean independent* which means that for all finite disjoint $F, G \subseteq \mathbb{N}$

$$\bigcap_{n\in F} A_n \cap \bigcap_{n\in G} B_n \neq \emptyset \ .$$

As an example consider the Rademacher functions (r_n) in L_∞. Let $A_n = [r_n = 1]$ and $B_n = [r_n = -1]$. These are Boolean independent and as a consequence,

$$\Big\|\sum a_i r_i\Big\|_\infty = \sum |a_i| \ .$$

We want to show that our (y_n) is also equivalent to the unit vector basis of ℓ_1, given the Boolean independence of $(A_n = [y_n > r+\delta],\ B_n = [y_n < r])$. By replacing (y_n) by $(-y_n)$ if necessary we may assume $r+\delta > 0$.

Then for all $(a_i)_1^n \subseteq \mathbb{R}$
$$\sum_1^n |a_i| \geq \Big\| \sum_{i=1}^n a_i y_i \Big\|_\infty \geq \frac{\delta}{2} \sum_1^n |a_i| \ .$$
To see this let $I_1 = \{i \leq n : a_i \geq 0\}$ and
$I_2 = \{i \leq n : a_i < 0\}$. Let
$$s_1 \in \bigcap_{I_1} A_i \cap \bigcap_{I_2} B_i \ , \qquad s_2 \in \bigcap_{I_2} A_i \cap \bigcap_{I_1} B_i \ .$$
Then we claim that
$$\sum_{i=1}^n a_i(y_i(s_1) - y_i(s_2)) \geq \delta \sum_{i=1}^n |a_i| \ .$$
Indeed

i)
$$\sum_{i=1}^n a_i y_i(s_1) \geq \sum_{I_1} |a_i|(r+\delta) - \sum_{I_2} |a_i| r$$
since
$$\sum_{I_2} a_i y_i(s_1) = -\sum_{I_2} |a_i| y_i(s_1) \geq \sum_{I_2} |a_i|(-r)$$
and similarly

ii)
$$-\sum_{i=1}^n a_i y_i(s_2) \geq -\sum_{I_1} |a_i| r + \sum_{I_2} |a_i|(r+\delta) \ .$$
Adding the two inequalities i) and ii), yields the claim.

Finally
$$\delta \sum_{i=1}^n |a_i| \leq \Big| \sum_1^n a_i(y_i(s_1) - y_i(s_2)) \Big| \leq 2 \Big\| \sum_1^n a_i y_i \Big\|_\infty \ .$$
Let us say $(A_n, B_n)_{n=1}^\infty$ has no convergent subsequence if for all $M \in [\mathbb{N}]^\omega$ there exists $s \in K$ with s belonging to infinitely many A_n's, $n \in M$, and also to infinitely many B_n's, $n \in M$.

One uses that (x_n) has no pointwise convergent subsequence on K to find $r \in [-1,1]$ and $\delta > 0$ with $(A_n = [x_n > r+\delta],\ B_n = [x_n < r])$ having no convergent subsequence (assume otherwise and inductively pass to convergent subsequences $(A_n, B_n)_{n \in M_i}$ for $(r_i, \delta_i)_i$ dense in $[-1,1] \times (0,1)$). Now Ramsey enters.

Lemma 2.7. *Let (A_n, B_n) be subsets of K with $A_n \cap B_n = \emptyset$ for all n. If (A_n, B_n) has no convergent subsequence then there exists $M \in [\mathbb{N}]^\omega$ so that $(A_n, B_n)_M$ is Boolean independent.*

Proof. Let $-A_n = B_n$ and
$$\mathcal{A} = \Big\{ L = (\ell_i)_1^\infty \in [\mathbb{N}]^\omega : \forall\ k\ ,\ \bigcap_{i=1}^k (-1)^i A_{\ell_i} \neq \emptyset \Big\} \ .$$

$\mathcal{A}$ is Ramsey (it is τ-closed) so there exits L with $[L] \subseteq \mathcal{A}$ or $[L] \subseteq \sim \mathcal{A}$. The latter case is impossible by hypothesis. Then $M = (\ell_{2i})_{1=1}^{\infty}$ works where $L = (\ell_i)$. Indeed if $F \cap G = \emptyset$, $F, G \subseteq M$ are finite, we can find $(\ell_{n_i})_{i=1}^{k}$ so that

$$\phi \neq \bigcap_{i=1}^{k} \varepsilon_i A_{\ell_{n_i}} \subseteq \bigcap_F A_n \cap \bigcap_G B_n \ .$$

□

Remark. From Rosenthal's ℓ_1 theorem and Proposition 2.4 we see that every X admits a 2-unconditional spreading model. Rosenthal [Ro2] proved that every X admits a 1-unconditional spreading model using the Borsuk-Ulam antipodal mapping theorem.

Another search for a "nice" subspace is weaker than that of the unconditional basic sequence search. If that were true then by James' work every X would contain either a reflexive subspace or an isomorph of c_0 or ℓ_1. The nonreflexivity would yield a sequence $(x_n) \subseteq B_X$ which either has no weak Cauchy subsequence, hence $\ell_1 \hookrightarrow X$ by Rosenthal's theorem, or a weak Cauchy sequence which is not weakly convergent. But this does not generally yield c_0 in X. We shall mention some more results in this setting.

If $\ell_1 \not\hookrightarrow X$ and we regard $X \subseteq C(K)$, $K = (B_{X^*}, \omega^*)$, compact metric, then by Theorem 2.6 every bounded sequence in X has a pointwise convergent subsequence on K. If X^* is separable it is easy to show that every $x^{**} \in X^{**}$, naturally, a function in $\ell_\infty(K)$, is in fact a ω^*-limit of a sequence in X, i.e., there exists $(x_n) \subseteq X$, $x_n \to x^{**}$ pointwise on K so in fact $x^{**} \in \beta_1(K)$, the first Baire class on K. But X^* need not be separable as witnessed by JT, the James tree space or JF, the James function space ([J2], [LS]).

It was shown in [OR] that $\ell_1 \not\hookrightarrow X$ iff every $x^{**} \in \beta_1(K)$ iff every x^{**} is a ω^*-limits of some sequence in X. So the topological nature of the functions $x^{**}|_K$ says something about the structure of X. In fact earlier Bessaga and Pełczyński [BP] had shown that $c_0 \hookrightarrow X$ iff there exits $x^{**} \in DBSC(K) \setminus X$. Here $DBSC$ is the difference of bounded semicontinuous functions. Further results of this kind appear in [HOR] and more on Baire-1 functions appear in [AK]. Rosenthal also proved a c_0 theorem, a companion to his ℓ_1 theorem, in the early 1990's [Ro3].

A nontrivial weak Cauchy sequence (non weakly convergent) has a subsequence (x_i) that is basic and dominates the summing basis,

$$\Big\| \sum a_i x_i \Big\| \geq c \Big\| \sum a_i s_i \Big\| \ .$$

(x_i) is called *strongly summing* if it is weak Cauchy basic and whenever $\sup_n \| \sum_1^n a_i x_i \| < \infty$ then $\sum_1^\infty a_i$ converges. Rosenthal proved: every nontrivial weak Cauchy sequence has either a strongly summing subsequence or admits a convex block basis equivalent to the summing basis.

Let's return to the problem of given an arbitrary space X to find an unconditional basic sequence within it. Of course as mentioned earlier this will prove to be impossible but let's follow what can be achieved. We can start with a normalized weakly null basic sequence (x_n) in X. By the Maurey-Rosenthal example we need not be able to find an unconditional subsequence. By the spreading model theory we can find an "asymptotically unconditional" subsequence (y_n). For some K, each n and $n \leq i_1 < \cdots < i_n$, $(y_{i_j})_1^n$ is K-unconditional. This is a notion of "partial unconditionality" of which there are several other versions.

Definition. (x_n) is *Schreier unconditional* if for some K and all $F \in S_1$ and $(a_i) \subseteq \mathbb{R}$,

$$\Big\| \sum_F a_i x_i \Big\| \le K \Big\| \sum a_i x_i \Big\| \ .$$

S_1 consists of all $F \in [\mathbb{N}]^{<\omega}$ so that $|F| \le \min F$. This is the *first Schreier class* of sets. Saying that (x_i) is Schreier unconditional is stronger than saying it is asymptotically unconditional.

It can be shown that if (x_i) is normalized weakly null then given $\varepsilon > 0$ some subsequence is $2+\varepsilon$-Schreier unconditional. Indeed we can presume (x_i) is nearly monotone and so the tail projections have norm close to 2:

$$\Big\| \sum_n^\infty a_i x_i \Big\| \le (2+\varepsilon) \Big\| \sum_1^\infty a_i x_i \Big\| \ .$$

Thus by a diagonal argument it suffices, given $n \in \mathbb{N}$, to find $(y_i) \subseteq (x_i)$ with

$$\Big\| \sum_F a_i y_i \Big\| \le (1+\varepsilon) \Big\| \sum a_i y_i \Big\|$$

if $|F| \le n$, $|a_i| \le 1$.
The idea is to choose (y_i) so that given F and $\sum a_i y_i$ as above and x^* with $\|x^*\| = 1$, $x^*(\sum_F a_i y_i) = \| \sum_F a_i y_i \|$ then there exists y^* with $\|y^*\| \lesssim 1$, $y^*(y_i) \approx x^*(x_i)$ for $i \in F$ and $\sum_{i \notin F} |y^*(y_i)| \approx 0$.

Then

$$\Big\| \sum_F a_i y_i \Big\| \approx y^* \Big(\sum_F a_i y_i \Big) \approx y^* \Big(\sum a_i y_i \Big) \le \Big\| \sum a_i y_i \Big\| \ .$$

Ultimately we will have $(y_i) = (x_{m_i})$. We choose (m_i) inductively.

To start let us say $x^* \in \mathcal{F} \subseteq B_{X^*}$ has *pattern* $\vec{b} = (b_1, \ldots, b_p)$ on $(\ell_i)_1^p \in [\mathbb{N}]^p$ if $x^*(x_{\ell_i}) = b_i$, $i \le p$. The nature of $\mathcal{F}$ will be made clear later. Set for $\vec{b}$ and $\delta > 0$,

$\mathcal{A} = \{L = (\ell_i)_0^\infty \in [\mathbb{N}]^\omega$: if there exists $x^* \in \mathcal{F}$ with pattern $\vec{b}$ on $(\ell_i)_1^p$ then there exists $y^* \in \mathcal{F}$ with pattern $\vec{b}$ on $(\ell_i)_1^p$ and $|y^*(x_{\ell_0})| < \delta\}$.

Passing to a subsequence of (x_i) we may assume W.L.O.G. that either for all $\ell_1 < \cdots < \ell_p$ there exists $x^* \in \mathcal{F}$ with pattern $\vec{b}$ on $(\ell_i)_1^p$ or this never happens. Now $\mathcal{A}$ is τ-closed hence Ramsey. We obtain $L \in [\mathbb{N}]^\omega$ with either $[L]^\omega \subseteq \mathcal{A}$ or $[L]^\omega \subseteq \sim \mathcal{A}$.

The latter case is impossible by Lemma 2.5.

If we repeat this for a finite ε-net of n-patterns we obtain $M^1 \in [\mathbb{N}]^\omega$, $M^1 = (m_i^1)$, so that we can, given $F \in [M^1]^n$ and $x^* \in \mathcal{F} = B_{X^*}$, preserve the values of x^*, up to ε, on F by some $y^* \in B_{X^*}$ and get $|y^*(x_{m_1^1})| < \delta_1$, provided $m_1^1 < F$.

We then split B_{X^*} into finitely many $\mathcal{F}_i^1$'s according to their values, up to δ_1, on $x_{m_1^1}$. We repeat the above argument for each $\mathcal{F}_i^1$ and $p = n, n-1$, and continue in this fashion.

Remarks. One consequence of the proof is that if (x_n) is normalized weakly null and $\varepsilon > 0$ there exists a subsequence (y_i) satisfying $(1-\varepsilon)\max|a_i| \le \|\sum a_i y_i\|$ for all (y_i), i.e., we can improve the lower c_0-estimate to nearly 1.

The reason this proof works for a fixed n but cannot be extended to yield an unconditional subsequence is that we would have to deal with infinitely many patterns. However there is a beautiful infinite partial unconditionality result of John Elton [Elt] which overcomes this problem by restricting the projection of $\sum a_i y_i$ onto sets $\sum_F a_i y_i$ where the coefficients $(a_i)_F$ are roughly of the same size, $\frac{|a_i|}{|a_j|} \le \delta^{-1}$.

Theorem 2.8. *For each $\delta > 0$ there exists $K(\delta) < \infty$ with the following property. Let (x_n) be a normalized weakly null sequence in a Banach space. There exists a subsequence (y_n) of (x_n) such that if $(a_i) \subseteq [-1,1]$ and $I \subseteq \{i : |a_i| \ge \delta\}$ then*

$$\Big\| \sum_I a_i y_i \Big\| \le K(\delta) \Big\| \sum a_i y_i \Big\| \ .$$

Sketch. Let's consider the final inequality we are trying to achieve. Let $\|y^*\| = 1$,

$$y^*\Big(\sum_I a_i y_i \Big) = \Big\| \sum_I a_i y_i \Big\| = \sum_I a_i y^*(y_i) \ .$$

By replacing vectors by their negatives if necessary we may assume for

$$I_{++} = \{ i \in I : a_i \ge 0,\ y^*(y_i) \ge 0 \}$$

$$y^* \sum_I a_i y_i \le 2 \sum_{I_{++}} a_i y^*(y_i) \le 2 \sum_{I_{++}} y^*(y_i) \ .$$

So the object will be to choose $(y_i) \subseteq (x_i)$ so that given $\varepsilon > 0$ and any set J and $y^* \in B_{X^*}$ we can find $x^* \in B_{X^*}$ with

$$\sum_J x^*(y_i)^+ \gtrsim \sum_J y^*(y_i)^+ \quad \text{and} \qquad \sum_{\{i : i \notin J \text{ or } x^*(y_i) < 0\}} |x^*(y_i)| < \varepsilon \ .$$

For then

$$\begin{aligned} \Big\| \sum_I a_i y_i \Big\| &\le 2 \sum_{I_{++}} y^*(y_i) \le 2 \sum_{I_{++}} x^*(y_i)^+ \\ &\le \frac{2}{\delta} \Big(\sum_{I_{++}} a_i x^*(y_i)^+ \Big) \le \frac{2}{\delta} \Big(x^* \Big(\sum a_i y_i \Big) + \varepsilon \Big) \\ &\le \frac{2}{\delta} \Big(\Big\| \sum a_i y_i \Big\| + \varepsilon \Big) \end{aligned}$$

If we restrict to $\| \sum a_i y_i \| \le M$ and achieve the above for $\sum_J x^*(y_i)^+ \ge B$ for a finite net of B's in $[0, M]$ we achieve the theorem with $K(\delta) \lesssim 2/\delta$. One uses a diagonal argument for the general statement.

So the key step is an argument like that used in Schreier unconditionality plus

Lemma 2.9. *Let $\mathcal{F} \subseteq B_{X^*}$, $B > 0$ and $\varepsilon > 0$. There exits $(y_i) \subseteq (x_i)$ so that if $n \in \mathbb{N}$ and $m_0 < m_1 < \cdots < m_n$ and if there exists $y^* \in \mathcal{F}$ so that $\sum_1^n y^*(y_{m_i})^+ \ge B$ then there exists $x^* \in \mathcal{F}$ with $\sum_1^n x^*(y_{m_i})^+ \ge B$ and $|x^*(y_{m_0})| < \varepsilon$.*

Proof. Let $\mathcal{A}_n = \{ M = (m_i)_0^\infty \in [\mathbb{N}]^\omega$: if there exists $y^* \in \mathcal{F}$ with $\sum_1^n y^*(x_{m_i})^+ \ge B$ then there exists $x^* \in \mathcal{F}$ with $\sum_1^n x^*(x_{m_i})^+ \ge B$ and $|x^*(x_{m_0})| < \varepsilon\}$.

Let $\mathcal{A} = \bigcap_n \mathcal{A}_n$ which is τ-closed and hence Ramsey. We obtain $L \in [\mathbb{N}]^\omega$ with either $[L]^\omega \subseteq \mathcal{A}$ or $[L]^\omega \subseteq\ \sim \mathcal{A}$.

If the second alternative occurs let $L = (\ell_i)$, $p \in \mathbb{N}$ and consider for $i \le p$

$$L_i = (\ell_i, \ell_{p+1}, \dots) \in\ \sim \mathcal{A} \ .$$

Thus $L_i \in \sim \mathcal{A}_{n_i}$ for some n_i and this is witnessed by some $y_i^* \in \mathcal{F}$, i.e.,

$$\sum_{j=1}^{n_i} y_i^*(x_{\ell_{p+j}})^+ \geq B \ , \text{ and } |y^*(x_{\ell_i})| \geq \varepsilon$$

$$\text{for any } y^* \in \mathcal{F} \text{ with } \sum_1^{n_i} y^*(x_{\ell_{p+j}})^+ \geq B \ .$$

Let i_0 satisfy $n_{i_0} = \inf\{n_i : i \leq p\}$. Then

$$\sum_{j=1}^{n_i} y_{i_0}^*(x_{\ell_{p+j}})^+ \geq B$$

for all $i \leq p$. Moreover $|y_{i_0}^*(x_{\ell_i})| \geq \varepsilon$ for all $i \leq p$. But this contradicts Lemma 2.5 if p is sufficiently large. □

An application of Theorem 2.8 is that if (x_n) is a normalized weakly null sequence then some subsequence (y_n) is either equivalent to the unit vector basis of c_0 or else for all $(a_i) \notin c_0$, $\lim_n \|\sum_1^n a_i y_i\| = \infty$. Indeed let $(y_n) \subseteq (x_n)$ satisfy the conclusion of 2.8 for $(y_n)_{n=k}^\infty$ and $\delta = 1/k$ for each k. Let $(a_i) \subseteq [-1,1]$ with $|a_i| \geq 1/k$ for $i \in M = (m_i) \in [\mathbb{N}]^\omega$ with $m_1 \geq k$ and

$$\sup_n \Big\| \sum_{j=k}^n a_j y_j \Big\| = K < \infty \ .$$

Then for each finite $F \subseteq M$

$$\Big\| \sum_{j\in F} a_j y_j \Big\| \leq K \cdot K(1/k) \ .$$

Thus for $f \in B_{X^*}$, $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_1^n |f(y_{m_j})| &= \sum_1^n \varepsilon_j f(y_{m_j}) \ , \quad \text{for } \ \varepsilon_j = \text{sign } f(y_{m_j}) \ , \\
&\leq k \sum_1^n \varepsilon_j |a_{m_j}| f(y_{m_j}) \\
&= k \Big[f\Big(\sum_F a_{m_j} y_{m_j} \Big) - f\Big(\sum_G a_{m_j} y_{m_j} \Big) \Big] \\
&\quad (\text{where } \ F = \{ j \leq n : \varepsilon_j |a_{m_j}| = a_{m_j} \} \\
&\quad \text{and } G = \{ j \leq n : \varepsilon_j |a_{m_j}| = -a_{m_j} \}) \\
&\leq 2kK \cdot K(1/k)
\end{aligned}$$

It follows that (y_{m_j}) is equivalent to the unit vector basis of c_0.

There is a very nice dual version of Elton's theorem due to Argyros, Merkourakis, and Tsarpalias [AMT].

Theorem 2.10. *For all $\delta > 0$ there exists $A(\delta) < \infty$ with the following property. Let (x_i) be a normalized weakly null sequence. There exists a subsequence $(y_i) \subseteq (x_i)$ satisfying:*

if $\sum |a_n| = 1$ and $\| \sum a_i y_i \| \geq \delta$ then
$\| \sum \varepsilon_i a_i y_i \| \geq A(\delta)\delta$ for all $\varepsilon_i = \pm 1$.

The proof follows the same sort of arguments as above. One uses if $\|x^*\| = 1$ and $x^*(\sum a_i y_i) \geq \delta$ then letting $I = \{i : x^*(y_i) \geq \delta/2\}$ one has $x^*(\sum_I a_i y_i) \geq \delta/2$. Then rather than preserving the positive ℓ_1 mass of x^* on I one seeks to preserve $x^*(y_i)^+ \geq \delta/2$.

Elton's theorem raises a beautiful open problem.

Problem 2.11. Does there exist $K < \infty$ so that for all $\delta > 0$ and all normalized weakly null (x_i) there exists $(y_i) \subseteq (x_i)$ satisfying : if $(a_i) \subseteq [-1, 1]$ and $I \subseteq \{i : |a_i| \geq \delta\}$ then $\|\sum_I a_i y_i\| \leq K\|\sum a_i y_i\|$.

One can do a bit better than $K(\delta) \leq c/\delta$ as we obtained above. In [DKK] it is shown that $K(\delta) \leq 16 \log_2(1/\delta)$ for $\delta \leq 1/4$. This follows from the trick of dividing, say for $\delta = 2^{-n}$, I into $I_0, I_1, \ldots, I_{n-1}$ where $I_j = \{i \in I : 2^{-i-1} \leq |a_i| < 2^{-i}\}$ and using for some j,

$$y^*\Big(\sum_{I_j} a_i y_i\Big) \geq \frac{1}{n} y^*\Big(\sum_I a_i y_i\Big) .$$

If Problem 2.11 has an affirmative answer then so does the following.

Problem 2.12. Does there exist $K < \infty$ so that every (x_i) equivalent to the unit vector basis of c_0 admits a K-unconditional subsequence?

These problems and more plus generalizations of the above partial unconditionality results are discussed at greater length in [DOSZ] and [LA-T]. For example one could ask if the $A(\delta)$ in Theorem 2.10 can be taken $A(\delta) \geq A > 0$ for all $\delta > 0$ and this is shown to be an equivalent problem to Problem 2.11. Problem 2.12 is nearly equivalent but it is not yet clear if it is equivalent.

Problem 2.11 has deep combinatorial aspects. In [DOSZ] it is shown $K(\delta) \geq 5/4$ for some δ which indicates pure combinatorics will not be enough.

Here are some other applications of Ramsey theory to Banach spaces, without proof.

Theorem 2.13. [KO1] *Assume every normalized weakly null sequence in X admits a subsequence equivalent to the unit vector basis of c_0. Then there exists $K < \infty$ so that every such sequence admits a subsequence K-equivalent to the unit vector basis of c_0.*

To prove this one need only obtain the upper estimate: $\|\sum a_i y_i\| \leq K \max |a_i|$. This was generalized in [KO2] and then more generally by Dan Freeman.

Theorem 2.14. [Free] *Let (v_i) be a normalized basis. Assume every normalized weakly null sequence in X admits a subsequence (y_i) which is dominated by (v_i), i.e., for some K*

$$\Big\|\sum a_i y_i\Big\| \leq K\Big\|\sum a_i v_i\Big\| .$$

Then some uniform K exists.

In every basic functional analysis course it is shown that in every X for every $\varepsilon > 0$ there exists $(x_i) \subseteq B_X$ with $\|x_i - x_j\| \geq 1$ if $i \neq j$. Indeed, choose inductively $(x_i, x_i^*) \subseteq S_X \times S_{X^*}$ with $x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0$ if $i < j$ for all i, j. Thus $\|x_i - x_j\| \geq x_i^*(x_i - x_j) = 1$. Using Ramsey theory we can do better.

Theorem 2.15. [EO] (see also [D]) *For every X there exists $\lambda > 1$ and $(x_i) \subseteq B_X$ so that $\|x_i - x_j\| > \lambda$ if $i \neq j$.*

Kryczka and Prus [KPr] proved that there exists $\lambda_0 > 1$ so that if X is not reflexive B_X contains a λ_0-separated sequence. It is unknown what maximum value λ_0 works for all nonreflexive spaces.

Ramsey theory is a vast subject. Let us mention a couple of additional results. More can be found, in addition to references given above, in [O1], [ATo],[GRS], [G3, G4].

Notation. $\vec{B} = (B_i)$ will denote an infinite sequence of finite subsets of $\mathbb{N}$ with $B_1 < B_2 < \cdots$. $\vec{B}_n$ will denote such a sequence of length n. $\vec{C} = (C_i) \leq \vec{B} = (B_i)$ if there exists $\vec{D} = (D_i)$ with for all i $C_i = \bigcup_{j \in D_i} B_j$. Similarly we define $\vec{C}_n \leq \vec{B}$. β_n will denote the set of all $\vec{B}_n$'s. β denotes the set of all $\vec{B}$'s.

Hindman-Milliken Theorem 2.16. [Hi], [Mi]) *Let $n \in \mathbb{N}$ and $\mathcal{A} \subseteq \beta_n$. Then there exists $\vec{B}$ so that either*

$$\{\vec{B}_n : \vec{B}_n \leq \vec{B}\} \subseteq \mathcal{A} \quad \text{or} \quad \{\vec{B}_n : \vec{B}_n \leq \vec{B}\} \subseteq \ \sim \mathcal{A}\ .$$

This theorem appears to potentially have application to block bases of a given basis and indeed we shall see one such result later. The obstacle is that while $\vec{B} = (B_i)$ might easily code the support of a block basis it does not distinguish the vectors.

There is also an infinite version of this theorem. We topologize β by the open base $F_{\vec{B}_n} = \{\vec{F} : \vec{F}_n = \vec{B}_n\}$ where $n \in \mathbb{N}$ and $\vec{B}_n$ are arbitrary.

Hindman-Milliken Theorem 2.17. *Let $\mathcal{A} \subseteq \beta$ be analytic. Then there exists $\vec{B}$ so that either*

$$\{\vec{D} : \vec{D} \leq \vec{B}\} \subseteq \mathcal{A} \quad \text{or} \quad \{\vec{D} : \vec{D} \leq \vec{B}\} \subseteq \ \sim \mathcal{A}\ .$$

There is a deep result that unifies many theorems in Ramsey theory including the ones we have stated and we end this section by stating it.

First we need some notation. We consider a finite alphabet and a variable v. We consider all infinite sequences $\vec{t} = (\vec{t}_n)$ of variable words $\vec{t}_n$, i.e., words formed from the alphabet and v containing at least one v each. We topologize this by an open base given by all sets

$$B_n(\vec{t}) = \{\vec{s} : \vec{s}|_n = \vec{t}|_n \ \text{ and } \ \vec{s} \text{ is a reduction of } \vec{t}\}$$

$\vec{s}$ is a *reduction* of $\vec{t}$, if $\vec{t} = (\vec{t}_1, \vec{t}_2, \ldots)$ and $\vec{s} = \langle \vec{t}_1(a_0)^\frown t_2(a_1)^\frown \cdots ^\frown t_{n_1}(a_{n_1}), \ldots \rangle$ where $\vec{t}_i(a)$ means replace v by a. $\mathcal{A}$ is *completely Ramsey* if $\forall\ n\ \forall\ \vec{t}\ \exists\ \vec{s} \in B_n(\vec{t})$ with $B_n(\vec{s}) \subseteq \mathcal{A}$ or $B_n(\vec{s}) \subseteq\ \sim \mathcal{A}$.

Theorem 2.18. [CSi] *$\mathcal{A}$ is completely Ramsey iff $\mathcal{A}$ has the Baire property.*

3. Looking for ℓ_p, distortion and unconditional basic sequences

One of our main themes is # 2: Given X find a "nice" $Y \subseteq X$. In the late 1960's and early 1970's it was conjectured that perhaps every X contained an isomorph of ℓ_p for some $1 \le p < \infty$ or c_0. In other words these spaces were a sort of atomic particle for every X.

V.D. Milman [Mil] investigated this problem and connected it with the notion of distortion. We begin with some definitions and easy observations.

Definition. Let $f : S_X \to \mathbb{R}$. f is *oscillation stable* if for all $Y \subseteq X$ and $\varepsilon > 0$ there exists $Z \subset Y$ with $\text{osc}(f, Z) \equiv \sup\{f(y) - f(z) : y, z \in S_Z\} < \varepsilon$.

Suppose f is uniformly continuous and not oscillation stable. Then there exists $Y \subseteq X$ and $\varepsilon_0 > 0$ so that for all $Z \subseteq Y$, $\text{osc}(f, Z) \ge \varepsilon_0$. It follows that there exists $W \subseteq Y$ and $a < b$ so that for all $Z \subseteq W$ $\exists$ $z_1, z_2 \in S_Z$ with $f(z_1) < a$ and $f(z_2) > b$. Let $A = \{w \in S_W : f(w) < a\}$. Then $d(A, \cdot)$ is a Lipschitz function on S_X which is not oscillation stable.

Thus every uniformly continuous $f : S_X \to \mathbb{R}$ is oscillation stable iff every Lipschitz $f : S_X \to \mathbb{R}$ is oscillation stable. Moreover not every such function is oscillation stable iff there exists $W \subseteq X$ and sets $A, B \subseteq S_W$ with $d(A, B) \equiv \inf\{\|a - b\| : a \in A,\ b \in B\} > 0$ so that $A \cap S_Z \ne \emptyset$ and $B \cap S_Z \ne \emptyset$ for all $Z \subseteq W\}$. If W is uniformly convex then one can use this to construct an equivalent norm on W which is not oscillation stable. One uses as a unit ball $\overline{co}(A \cup -A \cup \delta B_W)$ for sufficiently small δ.

Definition. $(X, \|\cdot\|)$ is λ-*distortable*, where $\lambda > 1$, if there exists an equivalent norm $|\cdot|$ on X (i.e., $I_d : (X, \|\cdot\|) \to (X, |\cdot|)$ is an isomorphism) so that for all $Y \subseteq X$

$$\sup\left\{\frac{|y|}{|z|} : y, z \in S_{(Y, \|\cdot\|)}\right\} > \lambda\ .$$

X is *arbitrarily distortable* if it is λ-distortable for all $\lambda > 1$.

If $|\cdot|$ distorts X then $|\cdot| : S_X \to \mathbb{R}$ is not oscillation stable. If $|\cdot|$ is an equivalent norm on $Y \subseteq X$ which distorts Y then it extends to an equivalent norm on X [Pe1] which necessarily is not oscillation stable. Thus from our earlier discussion we have

Proposition 3.1. *For a given X the following are equivalent.*

- i) *Not every Lipschitz $f : S_X \to \mathbb{R}$ is oscillation stable.*
- ii) *There exists $W \subseteq X$ and $A, B \subseteq S_W$ with $d(A, B) > 0$ and $A \cap S_Z \ne \emptyset$, $B \cap S_Z \ne \emptyset$ $\forall\ Z \subseteq W$.*
- iii) *(If X is uniformly convex) there exists a distortable $W \subseteq X$.*
- iv) *(If $X = \ell_p$, $1 < p < \infty$) X is distortable.*

Part iv) follows from iii) and the fact that ℓ_p is $1 + \varepsilon$-*minimal* for all $\varepsilon > 0$. This means that for all $X \subseteq \ell_p$, there exists $Y \subseteq X$ with $d(Y, \ell_p) < 1 + \varepsilon$. This in turn follows from the easy fact that every normalized block basis (b_i) of (e_i) in ℓ_p $(1 \le p < \infty)$ (or c_0) satisfies $\|\sum a_i b_i\|_p = (\sum |a_i|^p)^{1/p}$ (or $\|\sum a_i b_i\|_{c_0} = \sup |a_i|$) along with the selection in X of a normalized sequence which is a perturbation of such a (b_i).

Now at the time of Milman's work no distortable spaces were known to exist. In fact R.C. James [J3] had shown that

Theorem 3.2. *ℓ_1 and c_0 are not distortable.*

The idea of the proof for ℓ_1 is to assume $|\cdot|$ is an equivalent norm and $X \subseteq \ell_1$. By our previous discussion we may assume X is spanned by a $|\cdot|$-normalized block basis (x_i) satisfying $|\sum_n^\infty a_i x_i| \geq c_n \sum_n^\infty |a_i|$ for all (a_i) where $c_n > 0$ is maximal with this property. Thus $c_n \uparrow c$ and we choose n_0 with $c_{n_0} \approx c$. Let (b_n) be a $|\cdot|$-normalized block basis of $(x_i)_{n_0}^\infty$ with

$$b_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i \quad \text{and} \quad \sum_{i=k_{n-1}+1}^{k_n} |a_i| \approx c^{-1} .$$

Then

$$\Big|\sum \alpha_n b_n\Big| = \Big|\sum_n \sum_{i=k_{n-1}+1}^{k_n} \alpha_n a_i x_i\Big|$$

$$\gtrsim c \sum_n \sum_{i=k_{n-1}+1}^{k_n} |\alpha_n|\,|a_i| \approx \sum_n |\alpha_n| .$$

Thus given $\varepsilon > 0$ we obtained (b_n) with $\sum |\alpha_n| \geq |\sum \alpha_n b_n| \geq (1-\varepsilon) \sum |\alpha_n|$ and it follows that ℓ_1 is not distortable.

The c_0 theorem can be proved similarly. First by our remark earlier, after Schreier unconditionality, we can focus solely on the upper estimate; i.e., to choose a $|\cdot|$–normalized block basis (b_i) of (x_i), the unit vector basis of c_0, satisfying $|\sum a_i b_i| \leq (1+\varepsilon) \max_i |a_i|$. For fun we will use the Hindman-Milliken Theorem 2.16 to do this. If $B = (n_1, \ldots, n_k) \in [\mathbb{N}]^{<\omega}$ let $\eta(B) = \sup |\sum_1^k \varepsilon_i x_{n_i}|$ where the sup is taken over all $\varepsilon_i = \pm 1$. By Theorem 2.18 we can find, given $\delta > 0$, some $C > 0$ and $\vec{B} = (B_i)$ so that for all $\vec{D} \leq \vec{B}$, $|\eta(\vec{D}\,) - C| < \delta$. Assume, for simplicity, $\eta(\vec{D}\,) = C$ for $\vec{D} \leq \vec{B}$. Let (b_i) be the normalized block basis of (x_i) determined by this,

$$b_i = \sum_{j \in B_i} \varepsilon_j^i x_j / C \quad \text{where} \quad \varepsilon_j^i = \pm 1 \quad \text{is chosen to make } \|b_i\| = 1 .$$

It follows that for all n and $\varepsilon_i = \pm 1$, $\|\sum_1^n \varepsilon_i b_i\| \lesssim 1$ which yields the result. $\square$

Theorem 3.3. [Mil] *Let X be a Banach space such that either*

a) *for all $x \in X$ the function τ_x is oscillation stable; $\tau_x(y) \equiv \|x+y\|$ for $y \in X$,*

or

b) *every equivalent norm on X is oscillation stable (i.e., no subspace of X is distortable).*

Then there exists ℓ_p $(1 \leq p < \infty)$ or c_0 so that for all $\varepsilon > 0$, $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$ or $c_0 \overset{1+\varepsilon}{\hookrightarrow} X$.

Remarks. 1. For $x \in X$ and $y \in X$ define

$$\|y\|_x = \big\|\|x\|y\| + y\big\| + \big\|\|x\|y\| - y\big\| .$$

It is not hard to show that $\|\cdot\|_x$ is an equivalent norm on X. By either a) or b) each $\|\cdot\|_x$ is oscillation stable on X and these are the only norms needed in the proof.

2. If $(x_n) \subseteq X$ generates a spreading model $(\tilde{x}_n)$ then often $[(\tilde{x}_n)]$ does not embed into X. But if we have something stronger we are in business. First by a slight modification to

the proof of the existence of spreading models we can construct a *spreading model over* X: Given (x_n) normalized basic there exists $(y_n) \subseteq X$ so that for all $x \in X$ and $(a_i)_1^n$

$$\lim_{k_1\to\infty} \cdots \lim_{k_n\to\infty} \Big\| x + \sum_{i=1}^n a_i y_{k_i} \Big\| \equiv \Big\| x + \sum_{i=1}^n a_i \tilde{y}_i \Big\|$$

exists and forms a norm on $X \oplus [(\tilde{y}_n)]$. We say that (y_n) *doubly generates an* ℓ_p type over X if for all x and $(\alpha, \beta) \in S_{\ell_p^2}$

$$\lim_i \lim_j \|x + \alpha y_i + \beta y_j\| = \lim_i \|x + y_i\| = \|x + \tilde{y}_i\| \ .$$

An analogous definition holds for c_0 types. It is not hard to show that if (y_n) doubly generates an ℓ_p (or c_0) type over X then for all $\varepsilon > 0$ a subsequence of (y_n) is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p (or c_0). The object is to use the oscillation stability of all $\|\cdot\|_x$ to produce such a (y_n). The "p" involved comes from Krivine's theorem.

Theorem 3.4 ([K], and extensions [Le], [Ro4])**.**
a) *Given C, $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $n = n(C, \varepsilon, k)$ so that if $(x_i)_1^n$ is basic with basis constant less than C then there exists $p \in [1, \infty]$ and a normalized block basis $(y_i)_1^k$ of $(x_i)_1^n$ which is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p^k.*
b) *Let (x_n) be a normalized basic sequence with spreading model $(\tilde{x}_n)$. Then there exists $p \in [1, \infty]$ so that for all n and $\varepsilon > 0$ there exists an identically distributed block basis $(y_i)_1^\infty$ of (x_i) with spreading model $[(\tilde{y}_i)] \subseteq [(\tilde{x}_i)]$ so that $(\tilde{y}_i)_1^n$ is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p^n.*

We will not give the details of the proof of Milman's theorem. A somewhat different proof, based on an unpublished argument [HORS], appears in [BL]. Krivine's theorem was not available to Milman. We should note that finitely every norm or even uniformly continuous $f : S_X \to \mathbb{R}$ stabilizes, i.e., for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an n-dimensional $E \subseteq X$ with $\mathrm{osc}(f, S_E) < \varepsilon$ [MS]. This is closely related to Dvoretsky's famous theorm that $\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} X$ for all X, n, and $\varepsilon > 0$ [Dv].

Further Remarks.
τ_x is called a *degenerate type* on X; $\tau_x(y) = \|x + y\|$. A *type* $\tau_{(x_n)}$ on X generated by a bounded $(x_n) \subseteq X$ is given by $\tau_{(x_n)}(y) = \lim_n \|y + x_n\|$. Milman's work presaged a beautiful theorem of Krivine and Maurey [KM]. First Aldous [Ald] proved that every $X \subseteq L_1$ contains an isomorph of some ℓ_p (necessarily, $1 \le p \le 2$). Krivine and Maurey generalized the argument's essential features. X is called *stable* if for all bounded pairs of sequences $(x_n), (y_n)$ in X there exist subsequences $(x_n'), (y_n')$ so that $\lim_{n\to\infty} \lim_{m\to\infty} \|x_n' + y_n'\| = \lim_{m\to\infty} \lim_{n\to\infty} \|x_n' + y_m'\|$. They proved

Theorem 3.5. *Let X be stable. Then there exists $p \in [1, \infty)$ so that for all $\varepsilon > 0$ $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$.*

They also showed that the L_p ($1 \le p < \infty$) spaces are all stable. If one weakens the definition to (x_n) and (y_n) being weakly convergent one obtains the definition of *weakly stable*. In [ANZ] it is shown that weakly stable spaces contain ℓ_p or c_0, thus handling the c_0 case (c_0 is weakly stable but not stable). There is still no general criterion to determine if some ℓ_p or c_0 embeds into X.

As might be expected Milman raised the

Distortion Problems.

a) Is ℓ_p distortable for $1 < p < \infty$?
b) Is any X distortable?
c) Is every uniformly continuous $f : S_X \to \mathbb{R}$ oscillation stable?

A few years later B.S. Tsirelson [T] gave an example of a space not containing any isomorph of c_0 or ℓ_p ($1 \le p < \infty$). We define and discuss this space now. Following convention we actually define T, the dual space of Tsirelson's original example, as discussed by Figiel and Johnson [FJ].

We say that finite subsets of $\mathbb{N}$, $(E_i)_1^n$ are S_1-*admissible* if $n \le E_1 < \cdots < E_n$ (i.e., $E_1 < \cdots < E_n$ and $(\min E_i)_{i=1}^n \in S_1$). T is the completion $\overline{(c_{00}, \|\cdot\|)}$ where $\|\cdot\|$ satisfies for $x \in c_{00}$,

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| : (E_i)_1^n \text{ is } S_1\text{-admissible}\right\}\right) . \tag{3.1}$$

In this expression "Ex" is the restriction of x to E, $Ex(i) = x(i)$ if $i \in E$ and 0 otherwise.

First note that this "norm" is really an equation and not an explicit formula. One must show that such a norm exists. This implicit type of definition seems to be fundamental in constructing a space not containing any ℓ_p or c_0. The original Tsirelson space, T^*, was defined by taking its unit ball to be the smallest ball B containing all e_n's and having the property that if $(f_i)_1^n \subseteq B$ are S_1-admissible, i.e., $(\operatorname{supp} f_i)_1^n$ is S_1-admissible, then $\frac{1}{2}\sum_{i=1}^n f_i \in B$. Thus $\|x\|_T = \|x\|_B$ in our previous notation (B was "$\mathcal{F}$"). We will return to this point of view later.

To return to the existence of a norm satisfying (3.1) we state a more general result [OS1]. Let $\mathcal{N}$ denote the class of all norms $\|\cdot\|$ on c_{00} for which (e_i) is a normalized monotone basis for $\overline{(c_{00}, \|\cdot\|)}$ and which satisfy $\|\sum a_i e_i\| \ge \max|a_i|$. $P : \mathcal{N} \to \mathcal{N}$ is *order preserving* if $|\cdot| \le \|\cdot\| \Rightarrow P(|\cdot|) \le P(\|\cdot\|)$. Here $|\cdot| \le \|\cdot\|$ means for all x, $|x| \le \|x\|$.

Proposition 3.6. *Let $P : \mathcal{N} \to \mathcal{N}$ be order preserving. Then P admits a smallest fixed point. Thus there exists $\|\cdot\| \in \mathcal{N}$ with $\|\cdot\| = P(\|\cdot\|)$ and $\|\cdot\| \le |\cdot|$ whenever $P(|\cdot|) = |\cdot|$.*

The proof, which we omit, is not difficult using transfinite induction. The existence of $\|\cdot\|$ satisfying (3.1) follows by setting

$$P(\|\cdot\|)(x) = \|x\|_\infty \vee \sup\left\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| : \text{ is admissible}\right\}.$$

Theorem 3.7. *T has the following properties.*

(1) *The unit vector basis (e_i) is a 1-unconditional basis for T.*
(2) *T is reflexive.*
(3) *If $(\tilde{x}_i)$ is any spreading model of a normalized basic $(x_n) \subseteq T$ then for all (a_i)*

$$\frac{1}{2}\sum |a_i| \le \left\|\sum a_i \tilde{x}_i\right\| \le \sum |a_i| ,$$

i.e., $(\tilde{x}_i)$ is 2-equivalent to the unit vector basis of ℓ_1.
Moreover if $(x_i)_1^n$ is any normalized block basis of $(e_i)_n^\infty$ then

$$\left\|\sum_1^n a_i x_i\right\| \ge \frac{1}{2}\sum_1^n |a_i| .$$

(4) *T does not contain any isomorph of ℓ_p, $1 \le p < \infty$, or c_0.*
(5) *T is $2 - \varepsilon$ distortable for all $\varepsilon > 0$.*

Proof. 1) and the "moreover" part of (3) are easy from the definition of the norm. If some ℓ_p or c_0 embeds into T then some block basis of (e_i) must be equivalent to the corresponding unit vector basis of ℓ_p or c_0 and by 3) the only possibility is ℓ_1. Since ℓ_1 is not distortable, if we prove 5) we obtain 4). Then 2) follows since T has an unconditional basis but does not contain c_0 or ℓ_1. 3) follows since a basic sequence in a reflexive space must be weakly null.

We sketch the argument for 5). James' theorem that ℓ_1 is not distortable localizes.

Lemma 3.8. *Let $n \in \mathbb{N}$, $K > 1$ and let $(x_i)_1^{n^2}$ be a normalized basis K-equivalent to the unit vector basis of $\ell_1^{n^2}$. Then there exists a normalized block basis $(y_i)_1^n$ of $(x_i)_1^{n^2}$ which is $\sqrt{K}$-equivalent to the unit vector basis of ℓ_1^n.*

The idea is that if no subset $(x_i)_{i=(j-1)n+1}^{jn}$ for $1 \le j \le n$ is $\sqrt{K}$-equivalent to the unit vector basis of ℓ_1^n then each group admits a vector $y_j = \sum_{i=(j-1)n+1}^{jn} a_i x_i$ with $\|y_j\| = 1$ and $\sum_{i=(j-1)n+1}^{jn} |a_i| > \sqrt{K}$. An easy calculation yields $(y_j)_1^n$ is $\sqrt{K}$-equivalent to the unit vector basis of ℓ_1^n.

Note that by Lemma 3.8 and iteration if, for some K, $\ell_1^n \overset{K}{\hookrightarrow} X$ for all n then for all $\varepsilon > 0$, $\ell_1^n \overset{1+\varepsilon}{\hookrightarrow} X$ for all n.

Proof of 5). T is $2 - \varepsilon$ distortable.

We show that for some n large enough

$$\|x\|_n \equiv \|x\|_\infty \vee \sup \left\{ \frac{1}{2} \sum_{i=1}^n \|E_i x\| : (E_i)_1^n \text{ is } S_1\text{-admissible} \right\} .$$

$2 - \varepsilon$ distorts T. Clearly $\|\cdot\|_n \le \|\cdot\|$. We will show that for all $X \subseteq T$ there exists vectors $y, z \in X$ with $\|y\|, \|z\| \approx 1$, $\|y\|_n \approx 1/2$ and $\|z\|_n \approx 1$ with the approximations getting better as $n \to \infty$. Since X contains a perturbation of a block basis of (e_i) we may assume $X = [(x_n)]$ where (x_n) is a normalized block basis of (e_i). Since $(x_i)_{m+1}^{2m}$ is 2-equivalent to the unit vector basis of ℓ_1^m by (3), from Lemma 3.8, for all $\delta > 0$ and for all m there exists a normalized block $(y_i)_1^m$ of (x_i) which is $1 + \delta$-equivalent to the unit vector basis of ℓ_1^m. Let $y = \frac{1}{m} \sum_{i=1}^m y_i$ with $m \gg n$. Then $1 \ge \|y\| \ge \frac{1}{1+\delta}$ while $\|y\|_n \approx 1/2$. The idea behind the latter calculation is that there are many more y_i's than sets $(E_i)_1^n$ in any expression $\frac{1}{2} \sum_1^n \|E_i y\| = \|y\|_n$. Set $F = \{i \le m : |\{j : E_j y_i \ne 0\}| > 1\}$. These are the y_i's whose support is split by the E_j's. Then $|F| < n$ and so

$$\frac{1}{2} \sum_1^m \|E_j y\| \le \frac{1}{2} \sum_{j=1}^n \left\{ \left\| E_j \left(\frac{1}{m} \sum_{i \in F} y_i \right) \right\| + \left\| E_j \left(\frac{1}{m} \sum_{i \notin F} y_i \right) \right\| \right\}$$

$$\le \frac{1}{m} \Big\| \sum_F y_i \Big\| + \frac{1}{2} \frac{1}{m} \sum_{i \notin F} \|y_i\| < \frac{n}{m} + \frac{1}{2} < \frac{1}{2} + \delta \text{ for large } m .$$

The idea behind this calculation is that most of the y_i's are not split by $\frac{1}{2} \sum_1^n \|E_j(\cdot)\|$ and so using the triangle inequality the total mass they contribute is at most $1/2$. The few y_j's whose support is split could each contribute more to the norm but because $y = \frac{1}{m}\sum_1^m y_i$ the net contribution is n/m which is small.

The vector y we call an (ℓ_1^m, δ)-average. Next we consider n very large and form $z = \frac{2}{n}\sum_{i=1}^n y_i$ where each y_i is an $(\ell_1^{m_i}, \delta)$-average of block sequences in $\operatorname{span}(x_i)$, where m_i increases rapidly, m_{i+1} depends upon $\max(\operatorname{supp} y_i)$, and $n \le y_1 < y_2 < \cdots < y_n$. This has come to be called an RIS for rapidly increasing sequence of ℓ_1 averages and plays a key role in later results. Clearly $\|z\|_n \ge \frac{1}{2}(\frac{2}{n}\sum_{i=1}^n \|y_i\|) > \frac{1}{1+\delta}$. We next show that $\|z\| \lesssim 1$.

Let $\|z\| = \frac{1}{2}\sum_1^k \|E_i z\|$ where $(E_i)_1^k$ is S_1-admissible. Choose i_0 minimal so that $E_i x_{i_0} \ne 0$ for some i. Then there are relatively few E_i's relative to the length m_j of the average formed to obtain y_j if $j > i_0$. Hence

$$\|z\| \le \frac{2}{n}\Big(\frac{n}{2} + \varepsilon + 1\Big) = 1 + \frac{2\varepsilon + 2}{n}$$

for small enough δ by arguments like those used to compute $\|y\|_n$ in the first part. $\square$

This example ultimately had impact on both the distortion problem and the unconditional basic sequence problem but it took another 20 years to come to fruition. The next main step was due to Th. Schlumprecht [S1, S2].

First we note that by Milman's work T would have to contain a distortable subspace and as it turned out T itself is distortable. The proof shows that every $X \subseteq T$ contains two different types of vectors: ℓ_1^n-averages and averages of RIS averages. Even though it is very hard to explicitly compute norms in T, the definition of the norm could be used to distinguish these two types of vectors.

It remains open if T is more than $2 - \varepsilon$ distortable.

Definition. X is of *bounded distortion* if X is distortable but for some $\lambda > 1$, X is not λ-distortable.

Problem. Is T of bounded distortion? Is any X of bounded distortion?

Th. Schlumprecht constructed a space, now known as S, which is arbitrarily distortable.

Definition. S is the completion of c_{00} under the implicit norm $(f(n) \equiv \log_2(n+1))$

$$\|x\| = \|x\|_\infty \vee \sup\left\{\frac{1}{f(n)}\sum_{i=1}^n \|E_i x\| : 2 \le n \in \mathbb{N} \ \text{ and } \ E_1 < \cdots < E_n\right\}.$$

Because of the "damping factor," $\frac{1}{f(n)}$, S_1-admissibility is not needed.

Definition. X is *biorthogonally distortable* if there exists a sequence of equivalent norms $\|\cdot\|_n \le \|\cdot\|$ and $\varepsilon_n \downarrow 0$ so that we have the following. For all i_0 and $Y \subseteq X$ there exists $y \in S_{(Y, \|\cdot\|_{i_0})}$ with $\|y\|_j \le \varepsilon_{\min(i_0, j)}$ if $j \ne i_0$.

A biorthogonally distortable space is arbitrarily distortable but it is not known if an arbitrarily distortable space contains a biorthogonally distortable subspace. The arbitrary distortion comes from the norms $\|\cdot\|_n$.

Theorem 3.9. [S1, S2, AS2] *S is biorthogonally distortable. Moreover for all $X \subseteq S$ and $\varepsilon > 0$ there exists $Y \subseteq X$ with $d(Y, S) < 1 + \varepsilon$, and Y is complemented in S.*

So S is also minimal and, in fact, complementably minimal just like ℓ_p. We should perhaps note that T^* is minimal [CJT] while T contains no minimal subspace [CO].

The norms that biorthogonally distort S are given by $\|\cdot\|_{n_i}$ for some $n_i \uparrow \infty$ where

$$\|x\|_n = \|x\|_\infty \vee \sup\left\{ \frac{1}{f(n)} \sum_{i=1}^n \|E_i x\| : E_1 < \cdots < E_n \right\} .$$

The proof involves some delicate estimates. To get the minimal result one chooses an appropriate sequence of RIS averages $(x_i)_{i=1}^\infty$ in X, and shows it is $1+\varepsilon$-equivalent to the unit vector basis (e_i) of S. The fact that we have ℓ_1^n-averages and hence, RIS's, comes either from Krivine's theorem or an argument like that of Lemma 3.8. As with T, (e_i) is 1-unconditional and S is reflexive. Also, unlike T, (e_i) is 1-subsymmetric.

Let's first examine the relevance of S to the unconditional basic sequence problem. Gowers and Maurey [GM] observed the following. Note first that arbitrary distortion and biorthogonal distortion pass to subspaces.

Theorem 3.10. *Let X be biorthogonally distortable with a basis (e_i). Then for all $m \in \mathbb{N}$ there exists an equivalent norm $|\cdot|$ on X so that for all block basis (x_i) of (e_i) there exists a block basis $(y_i)_1^m$ of (x_i) with $d_b((y_i)_1^m, (s_i)_1^m) \le 10$.*

Here $(s_i)_1^m$ is the summing basis of length m. Thus a consequence of this theorem is that for all K, X can be renormed so that no sequence $(z_i)_1^\infty$ in X can be unconditional with constant less than K. The proof uses a Maurey-Rosenthal example type of argument. We may assume (e_i) is bimonotone.

Let $\|\cdot\|_i$ biorthogonally distort X (for some ε_i).

For $i \in \mathbb{N}$ set $X_i = (X, \|\cdot\|_i)$ and let ϕ be a coding:

$$\phi : \left\{ (x_i^*)_1^j : j \in \mathbb{N},\ x_1^* < \cdots < x_j^*,\ x_i^* \in X^*,\ x_i^*(e_\ell) \in \mathbb{Q} \text{ for all } i, \ell \right\} \xrightarrow{1-1} \mathbb{N} .$$

For $m \in \mathbb{N}$ set

$$\mathcal{F}^m = \left\{ \sum_1^m x_i^* : x_1^* < \cdots < x_m^*,\ x_1^* \in B_{X_1^*},\ x_{i+1}^* \in B_{X^*_{\phi(x_1^*,\ldots,x_i^*)}} \right\} .$$

The norm that works is $\|\cdot\|_{\mathcal{F}^m} \equiv |\cdot|$. In any block basis (x_i) of (e_i) we can find a block basis $(y_i)_1^m$ as follows.

Let $y_1 \in S_{X_1} \cap \langle x_i \rangle$ and choose $y_1^* \in S_{X_1^*}$, $y_1^*(y_1) = 1$ and $y_1^*(e_\ell) \in \mathbb{Q}$ for all ℓ and $\max(\operatorname{supp} y_1^*) \le \max(\operatorname{supp} y_1)$. Then choose y_2, $y_1 < y_2$, $y_2 \in S_{X_{\phi(y_1^*)}}$ and $y_2^* \in S_{X^*_{\phi(y_1^*)}}$ to norm it, $y_1^* < y_2^*$ and so on.

It follows that $|\sum_1^m a_i y_i| \ge \|\sum_1^m a_i s_i\|$. The upper estimate comes from the fact that if $\sum_1^j z_i^*$ is another element of $\mathcal{F}_m$ then $\sum_1^{j_0} z_i^* = \sum_1^{j_0} y_i^*$ for some j_0, $z_{j_0+1}^*$ and $y_{j_0+1}^*$ both belong to the same $S_{X^*_{\phi(y_1^*,\ldots,y_{j_0}^*)}}$ and after that $|z_\ell^*(y_i)| \approx 0$ for all i.

With a bit more care one can prove

Proposition 3.11. [OS2] *Let X be biorthogonally distortable with a basis (e_i). Then for all m and $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X with the following property. For all block bases (x_i) of (e_i) and all normalized monotone bases $(b_i)_1^m$ there exists a normalized block basis $(y_i)_1^m$ of (x_i) with $d_b((y_i)_1^m, (b_i)_1^m) < 1+\varepsilon$.*

Remark. We have seen that a normalized weakly null sequence has a subsequence which is $1+\varepsilon$-basic and hence the projection constant is less than $2+4\varepsilon$. In general one can not do much better than this. We can renorm S, or any biorthogonally distortable space, so that every block basis admits block vectors (y_1, y_2) with $d_b((y_1, y_2), (s_1, s_2)) < 1 + \varepsilon$. Since $\|s_1 - 2s_2\| = 1$ we get the best projection constant of any block basis is at least $2 - \varepsilon$.

So how did Gowers and Maurey proceed with this idea? S has an unconditional basis so even though we can make the constant arbitrarily large we cannot rid ourselves of the unconditional basis. The Maurey-Rosenthal coding argument must be built into the nature of the norm to handle all block bases and all K.

Theorem 3.12. [GM] *There exists a space X_{GM} not containing an unconditional basic sequence.*

The space is the completion of c_{00} under the following norm. We let $f(n) = \log_2(n+1)$ as in S and set

$$\|x\| = \|x\|_\infty \vee \sup \left\{ \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i x\| : 2 \le n,\ E_1 < E_2 < \cdots < E_n \text{ are all \textit{intervals} of integers} \right\}$$

$$\vee \sup \left\{ |\langle g, Ex \rangle| : g \text{ is a \textit{special function} and } E \text{ is an interval of integers} \right\}$$

The term special function needs explanation. Let Q denote all elements of $B_{c_0} \cap c_{00}$ having only rational values. Let $J = (j_i)$ be a lacunary subsequence of $\mathbb{N}$, $K = (j_{2i-1})$, $L = (j_{2i})$. Let σ be an injection from all block bases $(f_1, \ldots, f_n)$ in Q into L satisfying $\sigma(f_1, \ldots, f_n)$ is much larger than $\max(\operatorname{supp} f_n)$.

Now if $(Y, \|\cdot\|)$ is a space with $\|\cdot\| \in \mathcal{N}$, set

$$\mathcal{A}_m(Y) = \left\{ \frac{1}{f(m)} \sum_{i=1}^{m} f_i : f_1 < \cdots < f_n \ ,\ f_i \in B_{Y^*} \right\} .$$

Set

$$b_k(Y) = \left\{ (g_1, \ldots, g_k) : g_1 < \cdots < g_k \ ,\ g_i \in Q \ ,\ g_1 \in \mathcal{A}_{j_{2k}}(Y) \text{ and } g_{i+1} \in \mathcal{A}_{\sigma(g_1, \ldots, g_i)}(Y) \text{ for } i \le k-1 \right\} .$$

Let

$$B_k(Y) = \left\{ \frac{1}{\sqrt{f(k)}} \sum_{i=1}^{k} g_i : (g_i)_1^k \in b_k(Y) \right\} .$$

$B_k(Y)$ are the special functions.

Our norm is then

$$\|x\| = \|x\|_\infty \vee \sup \left\{ |g(x)| : g \in \mathcal{A}_m(X) \ ,\ m \ge 2 \right\} \vee \sup \left\{ |g(Ex)| : g \in B_k(X) \ ,\ k \ge 2 \ ,\ E \text{ an interval} \right\}$$

One needs to show some $\|\cdot\| \in \mathcal{N}$ satisfies this of course. It is then shown that one can always find (ℓ_1^n, ε) averages and RIS's. Given k and quite a few technical calculations one shows that given k one can produce a block sequence of RIS-averages, $(x_i)_1^k$, of any block basis of the basis (e_i) for X_{GM}, satisfying (essentially)

$$\Big\|\sum_1^k x_i\Big\| \geq \frac{1}{2}\frac{k}{\sqrt{f(k)}} \quad \text{while} \quad \Big\|\sum_1^k (-1)^i x_i\Big\| \leq \frac{2k}{f(k)} \ .$$

Now the proof actually yields more. Given block bases (y_i) and (z_i) of (e_i) one can choose the x_i's with great freedom. Namely $x_{2i} \in \langle y_j\rangle$ and $x_{2i+1} \in \langle z_j\rangle$. Thus we obtain $\bar y \in \langle y_j\rangle$, $\bar z \in \langle z_j\rangle$ with

$$\|\bar y + \bar z\| \geq \frac{1}{2}\ \frac{k}{\sqrt{f(k)}} \qquad \text{and} \qquad \|\bar y - \bar z\| \leq \frac{2k}{f(k)} \ .$$

From this we see

$$d(S_Y, S_Z) = \inf\Big\{\|y - z\| : y \in S_Y,\ z \in S_Z\Big\} = 0$$

for all $Y, Z \subseteq X_{GM}$. In other words X_{GM} is hereditarily indecomposable.

In the next definition X need not be separable.

Definition. An infinite dimensional X is H.I. (*hereditarily indecomposable*) if for all $Y \subseteq X$, if $Y = Y_1 \oplus Y_2$ then $\inf(\dim Y_1, \dim Y_2) < \infty$.

In other words Y contains no nontrivial complemented subspaces. Of course an H.I. space cannot contain an unconditional basic sequence. For if (x_i) is unconditional $[(x_i)] = [(x_{2i})] \oplus [(x_{2i-1})]$. We also have the remarkable,

Theorem 3.13. [GM]

a) *An H.I. space is not isomorphic to its hyperplane nor to any proper subspace.*
b) *If $T : X_{GM} \to X_{GM}$ is a bounded linear operator then for some $\lambda \in \mathbb{R}$, $T = \lambda I + S$ where I is the identity and S is* strictly singular *(S is not an isomorphism restricted to any $Y \subseteq X_{GM}$).*

Problem. Does there exist X so that all bounded linear operators on X have the form $\lambda I + K$ where K is compact?

This is not true for X_{GM} [AS1]. They showed also that $\ell_\infty \hookrightarrow B(X_{GM})$, the space of bounded linear operators on X_{GM}.

Problem. For all X does $\ell_\infty \hookrightarrow B(X)$?

Once X_{GM} was constructed many other H.I. spaces followed. In fact they are everywhere.

Theorem 3.14. [AF], [AT2] *Every X contains either an isomorph of ℓ_1 or is a quotient of an H.I. space.*

Of course if X contained ℓ_1 and $Q : Y \to X$ was a quotient map then Y contains ℓ_1 (let $Qy_i = e_i$, the unit vector basis of ℓ_1, $\sup_i \|y_i\| < \infty$, so $[(y_i)] \sim \ell_1$).

Every quotient of X_{GM} is also H.I. [Fr].

T and S could be called, roughly, first level spaces and X_{GM} is a second level space by virtue of the definition of the norm. A third level norm was used in [OS1] to prove

Theorem 3.15. *There exists a reflexive space X with a basis (e_i) satisfying the following. For all n, $\varepsilon > 0$ and normalized monotone bases $(b_i)_1^n$ and for all block bases (x_i) of (e_i) there exists a block basis $(y_i)_1^n$ of (x_i) with $d_b((y_i)_1^n, (b_i)_1^n) < 1 + \varepsilon$.*

This can be viewed as the ultimate block basis version of the Maurey-Rosenthal example.

A weaker version of "Does every X contain an unconditional basic sequence?' is another # 2 type question. Does every X contain an isomorph of c_0, ℓ_1 or a reflexive space? By James' earlier result this of course holds if X contains an unconditional basic sequence. Gowers [G1] constructed a counter example to this (see also [AM] for more examples).

Theorem 3.16. *There exists a space X so that $\ell_1 \not\hookrightarrow X$ and yet for all $Y \subseteq X$, Y^* is not separable.*

The following problem remains open.

Problem. Assume that Y^* is nonseparable for all $Y \subseteq X$. Is ℓ_1 finitely representable in X?

A space Z is *finitely representable* in X if for all finite dimensional $F \subseteq Z$ and $\varepsilon > 0$ there exists $G \subseteq X$ with $d(G, F) < 1 + \varepsilon$. For $Z = \ell_1$ we need only take $F = \ell_1^n$, $n \in \mathbb{N}$.

Remarks. As we said there are now many H.I. spaces that have been constructed, many due to S. Argyros and his co-authors who have done some remarkable work. We quickly mention a *few* of their results.

1. [AD] There exists an asymptotic ℓ_1 H.I. space. (Asymptotic ℓ_p spaces will be discussed more later.)
2. [A] If every separable H.I. space embeds into X then so does $C[0,1]$ (more about universal spaces will be discussed later).
3. [AL-AT] A nonseparable reflexive space not containing an unconditional basic sequence is constructed.
4. [AT2] There exists X with X, X^* separable and H.I., X^{**} is nonseparable and H.I. Moreover the bounded operators on X^{**} all have the form $\lambda I_d + R$ where R has separable range.
5. [AT1] A reflexive X which is unconditionally saturated (every $Y \subseteq X$ contains an unconditional basic sequence) and is indecomposable is constructed.
6. [AM] A reflexive indecomposable space is constructed with every basic sequence admitting an unconditional subsequence.

It is worth noting that H.I. spaces cannot be too big. For the next few results we drop our convention that $X, Y, \dots$ are separable.

Proposition 3.17. [AT2] *If X is H.I. then $X \hookrightarrow \ell_\infty$.*

We will sketch the proof which gives us a chance to say a bit more about H.I. spaces.

Lemma A. *Let X be H.I. and let $T : X \to Y$ be a bounded linear operator. Then exactly one of the following holds.*

i) *T is strictly singular.*

ii) *$\dim(\operatorname{Ker} T) < \infty$ and $X = Z \oplus \operatorname{Ker} T$ with $T|_Z$ being an isomorphism.*

If $Y = X$ and T is 1–1 then T is either strictly singular or an onto isomorphism.

Proof. Let $\|T\| = 1$. If i) fails there exists an infinite dimensional $W \subseteq X$ and $\varepsilon > 0$ with $\|Tw\| \geq \varepsilon$ for all $w \in S_W$. If ii) also fails there exists an infinite dimensional $Z \subseteq X$ with $\|Tz\| < \varepsilon/2$ for all $z \in S_Z$. Thus for $w \in S_W$, $z \in S_Z$, $\|w - z\| \geq \|Tw - Tz\| > \varepsilon/2$ so X is not H.I.

This also yields the last statement except for the "onto" part which follows from [GM] since X is H.I. (and so is not isomorphic to a proper subspace). □

Lemma B (Milman)**.** *X is H.I. iff for all infinite dimensional $Y \subseteq X$, $\varepsilon > 0$ and $\mathcal{F} \subseteq B_{X^*}$ which ε-norms Y,*

$$\mathcal{F}_\perp \equiv \{x \in X : f(x) = 0 \quad \text{for all} \quad f \in \mathcal{F}\}$$

is finite dimensional.

Proof. We prove the "if" direction. Let $Y \oplus Z \subseteq X$ with $\dim Y = \infty$. Let P_Y be the projection of $Y \oplus Z$ onto Y with Kernel Z and $\varepsilon = \|P_Y\|^{-1}$. For $y^* \in \varepsilon B_{Y^*}$, extend this to $y^* \oplus 0 \in B_{(Y \oplus Z)^*}$ and let $\widetilde{y^*}$ be a Hahn-Banach extension to an element of B_{X^*}. Let $\mathcal{F}$ be all such $\widetilde{y^*}$'s. $\mathcal{F}_\perp \supseteq Z$ so $\dim Z < \infty$. □

Proof of Proposition 3.17. Let $Y \subseteq X$ be separable and infinite dimensional. Let $D \subseteq S_{X^*}$ be countable and $\frac{1}{2}$-norm Y. Then $D_\perp$ is finite dimensional by Lemma B. Choose a finite $F \subseteq S_{X^*}$ so that $(D \cup F)_\perp = \{0\}$. Write $D \cup F = (x_n^*)_{n \in \mathbb{N}}$. Let $Tx = (x_n^*(x)) \in \ell_\infty$. T is 1–1 and $T|_Y$ is an isomorphism so T is an into isomorphism by Lemma A. □

Well, the type # 2 questions: the unconditional basic sequence problem and the c_0, ℓ_1 or reflexive subspace problem have negative answers. One must assume some additional structure to achieve positive answers. We shall briefly discuss some more of these results.

Following [KM] some investigation was conducted with types to see if more could be obtained. Let Υ denote the set of all types τ on X. Recall $\tau(y) = \lim_n \|y + x_n\|$ for some bounded $(x_n) \subseteq X$. The *uniform metric* ρ_u is defined on Υ by

$$\rho_u(\tau, \sigma) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{\|y\| \leq k} |\tau(y) - \sigma(y)| \ .$$

It can be shown that (Υ, ρ_u) is separable if X is stable, so it is natural to ask what this property itself yields. Haydon and Maurey proved

Theorem 3.18. [HM] *Assume the set of types on X is separable for the uniform metric. Then X contains either a reflexive subspace or an isomorph of ℓ_1.*

The types of Tsirelson's space T are separable in the uniform metric [O2] so there is no chance of expanding this to get some ℓ_p. But one obtains some ℓ_p if the types from B_X are compact [HORS].

There is a recent paper of Argyros, Dodos and Kanellopoulos [ADK] with some spectacular positive results.

Theorem 3.19. [ADK]

1) *Let X^* be nonseparable. Then X^{**} contains an unconditional family of size $|X^{**}|$.*
2) *For all X one of the following holds.*
    a) *X is saturated with reflexive subspaces.*
    b) *X^{**} admits an unconditional family of size $|X^{**}|$.*
    c) *X^{***} admits an unconditional family of size $|X^{***}|$.*
3) *Every dual Banach space Y^* has a separable quotient X.*

This partially solves a famous problem

Problem. Does every infinite dimensional Banach space have an infinite dimensional separable quotient?

Equivalently, [JR], can every space be written as $\overline{\cup_n X_n}$ where $X_1 \subsetneqq X_2 \subsetneqq \cdots$ are closed subspaces (and for this we again drop the convention that X_n denotes a separable space)?

4. Distortion and Gowers Dichotomy Theorem

We have seen that distortable spaces exist, and moreover, every X either contains a distortable subspace or an isomorph of some ℓ_p $(1 \le p < \infty)$ or c_0.

Two main problems remain.

1. To classify which X, if any, have the property that every uniformly continuous $f : S_X \to \mathbb{R}$ is oscillation stable (for all $\varepsilon > 0$, $Y \subseteq X$ there exists $Z \subseteq Y$ with $\mathrm{osc}(f, S_Z) < \varepsilon$).
2. Is ℓ_2 distortable? Is ℓ_p distortable for $1 < p < \infty$?

Recall that ℓ_1 and c_0 are not distortable.

We have also seen that ℓ_2 is distortable (this also holds for ℓ_p, $1 < p < \infty$) iff there exist sets $A, B \subseteq S_{\ell_2}$ with $d(A, B) > 0$ and for all $X \subseteq \ell_2$, $A \cap S_X \neq \emptyset$ and $B \cap S_X \neq \emptyset$. How could we find such sets A and B. If we walked over S_{ℓ_2} how could we decide which vectors to put in A and which in B? If we stood at $x \in S_{\ell_2}$ it would look no different than standing at any other $y \in S_{\ell_2}$. Could the answer be different for different ℓ_p's? No by virtue of the Mazur map, M_p.

$$M_p : S_{\ell_1} \to S_{\ell_p} \ \text{ is given by } \ M_p((a_i)_{i=1}^\infty) = (\mathrm{sign}\ a_i |a_i|^{1/p})_{i=1}^\infty \ .$$

M_p is a uniform homeomorphism between the two unit spheres [Rib] and thus $f_{p,q} \equiv M_q M_p^{-1} : S_{\ell_p} \to S_{\ell_q}$ is a uniform homeomorphism (1–1, onto and the map and its inverse are uniformly continuous). Moreover $f_{p,q}$ and $f_{p,q}^{-1}$ preserve block subspaces (subspaces spanned by block bases). It follows that if some ℓ_p, $1 \le p < \infty$ admits sets $A, B \subseteq S_{\ell_p}$ with $d(A, B) > 0$ and $A \cap X \neq \emptyset$, $B \cap X \neq \emptyset$ for all $X \subseteq \ell_p$ then this holds for all $p \in [1, \infty)$. For example if such sets exist in S_{ℓ_1} then, for small enough ε, $((M_p(A))_\varepsilon, (M_p(B))_\varepsilon)$ form such a pair in S_{ℓ_p}. Here, for $C \subseteq S_{\ell_p}$, $C_\varepsilon \equiv \{x \in S_{\ell_p} : d(x, C) < \varepsilon\}$. In particular we see that if ℓ_2 is distortable then not every uniformly continuous $f : S_{\ell_1} \to \mathbb{R}$ is oscillation stable (let $f = d(A, \cdot)$).

Now we do have distortion in some spaces such as T or S. We can indeed identify different types of vectors in these spaces (ℓ_1^n-averages and averages of RIS's). So to solve the distortion problem the rough idea is to generalize the Mazur map to more spaces and use the existence of A and B in some space to transfer it back to ℓ_2.

The generalized Mazur map was prepared for use in [Gil], [Lo], although they did not focus on the properties of the map as such. First we define the *entropy* map

$$E_X : (\ell_1 \cap c_{00}) \times X \to [-\infty, \infty)$$

where X is a space with a normalized 1-unconditional basis (e_i). E_X is given by

$$E_X(h, x) = \sum_i |h_i| \log |x_i|$$

$(0 \log 0 \equiv 0)$.

$F_X : S_{\ell_1} \cap c_{00} \to S_X$ is given by $F_X(h) =$ the unique $x \in S_X$ so that if $x \in \sum x_i e_i$ then

i) $E(h, x) \ge E(h, y)$ for all $y \in S_X$
ii) $\mathrm{supp}\, h = \mathrm{supp}\, x$
iii) $\mathrm{sign}\ h_i = \mathrm{sign}\ x_i$ for all i

Of course such an x must be shown to exist. Now if X is uniformly convex and uniformly smooth one can show that F_X extends to a uniform homeomorphism of S_{ℓ_1} onto S_X. If $X = \ell_p$ then $F_X = M_p$.

With some additional tricks we obtain

Theorem 4.1. [OS2] *Let X have an unconditional basis. Then S_X and S_{ℓ_1} are uniformly homeomorphic iff c_0 is not finitely representable in X.*

The "only if" direction is due to Enflo [E2]. More general results exist (see [BL]).

Theorem 4.2. [OS2] *For $1 < p < \infty$, ℓ_p is biorthogonally distortable. More precisely, there exists $\varepsilon_i \downarrow 0$ and sets $\mathcal{A}_k \subseteq S_{\ell_p}$ and $\mathcal{A}_k^* \subseteq B_{\ell_q}(\frac{1}{p} + \frac{1}{q} = 1)$ for $k \in \mathbb{N}$ so that*

i) *for all $x \in \mathcal{A}_k$ there exists $x^* \in \mathcal{A}_k^*$ with $x^*(x) > 1 - \varepsilon_k$*
ii) *for all $x \in \mathcal{A}_k$ and $x^* \in \mathcal{A}_\ell^*$ with $\ell \neq k$ $|x^*(x)| < \varepsilon_{\min(k,\ell)}$*
iii) *for all $\varepsilon > 0$ and k, $(\mathcal{A}_k)_\varepsilon \cap Y \neq \emptyset$ for all $Y \subseteq \ell_p$.*

If $p = 2$ then $\mathcal{A}_k = \mathcal{A}_k^*$. Moreover the sets $\mathcal{A}_k$ and $\mathcal{A}_k^*$ are spreading and 1-unconditional (i.e., $x \in \mathcal{A}_k$ iff $|x| \in \mathcal{A}_k$).

The biorthogonal distortion of ℓ_p comes from the norms $\|x\|_n = \|x\|_{\mathcal{A}_n^*} \vee \varepsilon_n \|x\|$.

We outline the steps in the proof. First we recall from [S1] that there exist integers $p_\ell \uparrow \infty$ so that setting

$$C_k = \{x \in S : x \text{ is an RIS average of length } p_k\}$$

$$C_k^* = \left\{ x^* \in S^* : x^* = \frac{1}{f(p_k)} \sum_1^{p_k} x_i^* \ , \ (x_i^*)_1^{p_k} \text{ is a block basis of } B_{S^*} \right\}$$

then (C_k, C_k^*) satisfy i)–iii) above with respect to S. Of course these must be precisely quantified.

We then define sets in S_{ℓ_1},

$$B_k = \left\{ \frac{x_k^* \circ x_k}{|x_k^*|(|x_k|)} : x_k^* \in C_k^* \ , \ x_k \in C_k \ , \ |x_k^*|(|x_k|) = \|x_k^* \circ x_k\|_1 \geq 1 - \varepsilon_k \right\} .$$

Here $x_k^* \circ x_k = (x_k^*(i) x_k(i))_{i=1}^\infty$.

Each B_k is 1-unconditional and spreading. Moreover (this takes work) $(B_k)_\varepsilon \cap Y \neq \emptyset$ for all $Y \subseteq \ell_1$. This is done using the maps E_{S^*} and F_{S^*}.

Let's see how to proceed in ℓ_2 (for simplicity). Set

$$\mathcal{A}_k = \{x \in S_{\ell_2} : |x|^2 \in B_k\} = \mathcal{A}_k^* \ .$$

Thus $\mathcal{A}_k = M_2(B_k)$, so $\mathcal{A}_k$ satisfies iii), and also i). To see ii) let

$$\begin{aligned} & x_k \in \mathcal{A}_k, \ x_\ell \in \mathcal{A}_\ell \text{ with } k \neq \ell \\ \text{and } & |x_k|^2 = (x_k^* \circ x_k)/|x_k^*(|x_k|)| \\ \text{and } & |x_\ell|^2 = (x_\ell^* \circ x_\ell)/|x_\ell^*|(|x_\ell|), \end{aligned}$$

be as in the definition of B_k and B_ℓ, $\lambda = (1 - \varepsilon_1)^{-1}$.

Then

$$\begin{aligned}\langle |x_k|, |x_\ell|\rangle &\le \lambda \sum_j |x_k^*(j)x_k(j)x_\ell^*(j)x_\ell(j)|^{1/2}\\ &\le \lambda\Big(\sum_j |x_k^*(j)x_\ell(j)|\Big)^{1/2}\Big(\sum_j |x_\ell^*(j)x_k(j)|\Big)^{1/2}\\ &\le \lambda\langle |x_k^*|, |x_\ell|\rangle^{1/2}\langle |x_\ell^*|, |x_k|\rangle^{1/2}\\ &\le \lambda\varepsilon_{\min(k,\ell)}\ .\end{aligned}$$

Remarks. Our original argument was "Maureyed". Maurey also extended the proof to show that the sets B_k could be taken to be symmetric; $(x(i)) \in B_k \Leftrightarrow (x_{\pi(x)}) \in B_k$ for all permutations π of $\mathbb{N}$ [M1].

Maurey [M2] also proved that if X has an unconditional basis and ℓ_1 is not finitely representable in X then X contains an arbitrarily distortable subspace.

Theorem 4.3. [G2] *Every uniformly continuous $f : S_{c_0} \to \mathbb{R}$ is oscillation stable.*

The proof also appears in [BL] and involves an ingenious use of idempotent ultrafilters and the special property of c_0 that if x and y are close on all coordinates, say $\le \varepsilon$, then $\|x-y\| \le \varepsilon$. The proof was motivated by an ultrafilter proof of Hindman's theorem due to Glazer.

Thus not every Lipschitz $f : S_{\ell_1} \to \mathbb{R}$ is oscillation stable but for c_0 the picture is different.

We combine the theorems above with Milman's Theorem 3.3 to get the following

Theorem 4.4.

a) *X does not contain a distortable subspace iff for all $Y \subseteq X$ either c_0 or ℓ_1 embeds into Y.*
b) *Every $f : S_X \to \mathbb{R}$ is oscillation stable iff X is c_0-saturated (i.e., for all $Y \subseteq X$, $c_0 \hookrightarrow Y$).*

In our discussion of the unconditional basic sequence problem we saw that Ramsey theory was quite useful in obtaining results via stabilization of some property in terms of subspaces given by subsequences of a basis.

Since X_{GM} exists there can not be a Ramsey theorem that could succeed in always finding an unconditional basic sequence. Since ℓ_2 is distortable we know that even in a nice space we *cannot* say: if we finitely color S_{ℓ_2} then some subspace is, up to an ε, monochromatic. This only works for c_0-saturated spaces. Of course for seeking unconditional bases, or more generally a "nice" subspace, one needs only check among all block bases of a given basis.

It turns out that there is indeed a sort of "block Ramsey" theorem due to Gowers [G3] and [G4] but it is phrased, roughly, in terms of either you get a monochromatic block subspace or you have a winning strategy for the alternative. An application is Gowers dichotomy theorem.

First we need some notation. Assume X has a basis (e_i) and let $\Sigma = \Sigma(X)$ consist of all finite block bases of (e_i) in B_X. Let $\sigma \subseteq \Sigma$ and we consider a 2-player game. Player (S) (for subspace) chooses a block subspace X_1 of X, $X_1 = [(x_i')]$ when (x_i') is a block basis of (e_i). Then player (V) (for vector) chooses $x_1 \in B_{\langle x_i'\rangle}$ and plays alternate in this fashion. (V) wins if at some point $(x_1, x_2, \ldots, x_n) \in \sigma$.

If Y is a block subspace of X then σ is called *strategically large* for Y if (V) has a winning strategy for the game played solely in Y.

Let $\bar{\delta} = (\delta_i)_1^\infty$, $\delta_i > 0$. We set

$$\sigma_{\bar{\delta}} = \left\{ (x_i)_1^n \in \Sigma : \exists\ (y_i)_1^n \in \sigma \ \text{ with } \ \|x_i - y_i\| \le \delta_i \ \text{ for } \ i \le n \right\} .$$

This is the familiar "fattening" one needs when analysis meets Ramsey theory. Finally we say σ is *large for* X if for all block subspaces W of X some further block basis of W, $(w_i)_1^n \in \sigma$.

Theorem 4.5. [G3], [G4] *Let $\sigma \subseteq \Sigma(X)$ be large for X and $\bar{\delta} = (\delta_i)$, $\delta_i > 0$. Then there exists a block subspace Y of X so that $\sigma_{\bar{\delta}}$ is strategically large for Y.*

So given any X and $\sigma \subseteq \Sigma(X)$ we can find Y so that either $\sigma \cap \Sigma(Y) = \emptyset$ or $\sigma_{\bar{\delta}}$ is strategically large for Y.

This beautiful theorem has an important application which we now present.

Theorem 4.6 (Gowers' Dichotomy Theorem)**.** *Every X contains Y which either has an unconditional basis or is H.I.*

Sketch of proof. The proof is based upon the fact that we may assume X has a basis (e_i) and if X contains no unconditional basic sequence then for all N and all block bases of (e_i) one can find a further block basis $(y_i)_1^n \in \Sigma$ with

$$\Big\| \sum_1^n (-1)^i y_i \Big\| > N \Big\| \sum_1^n y_i \Big\| .$$

We let σ_N be all such blocks.

So $\sigma_{N/2}$ is large for X for all N. Thus given N we can find a block subspace Y of X so that $(\sigma_{N/2})_{\bar{\delta}}$ is strategically large for Y. Using $(\sigma_{N/2})_{\bar{\delta}} \subseteq \sigma_N$, for an appropriate $\bar{\delta}$, and a diagonal argument we obtain Z so that σ_N is strategically large for Z for all N. Then just as in the proof that X_{GM} is H.I. we obtain Z is H.I. $\square$

Theorem 4.5 seems to be not that easy to use. It is obvious that given σ one can find Y with either $\Sigma(Y) \subseteq \sigma^c$ or for all block subspaces Z of X, $\Sigma(Z) \cap \sigma \ne \emptyset$.

There is also an infinite version of Theorem 4.5 due to Gowers. One puts a suitable complete metric on $\Sigma_\infty(X)$ (roughly, the topology of pointwise convergence).

Theorem 4.7. [G3], [G4] *If $\sigma \subseteq \Sigma_\infty(X)$ is analytic and large for X then for all $\bar{\delta}$, X has a block subspace Y with $\sigma_{\bar{\delta}}$ strategically large for Y.*

Gowers used this theorem to prove

Theorem 4.8. *Let X have an unconditional basis. Then there exists a block subspace Y which is either quasi-minimal or no two subspaces generated by disjointly supported block bases are isomorphic.*

Y is *quasi-minimal* if for all $W, Z \subseteq Y$ there exists $U \subseteq W$ and $V \subseteq Z$ with U isomorphic to V. As an example, T is quasi-minimal (see e.g., [CS]).

An important application of Gowers' dichotomy theorem was the final solution to the famous homogeneous Banach space problem.

It had been shown [K, T-J1] that if X is homogeneous and not isomorphic to ℓ_2 then X contains a subspace without an unconditional basis.

Theorem 4.9 ([G3], [K, T-J1], [T-J1]). *Assume X is homogeneous (i.e., $X \sim Y$ for all $Y \subseteq X$). Then $X \sim \ell_2$.*

Indeed let X be homogeneous and choose $Y \subseteq X$ with either

a) an unconditional basis or

b) Y is H.I.

If $X \not\sim \ell_2$ then both a) and b) are impossible, the first by the [K, T-J1] result and the second since an H.I. space is not isomorphic to any proper subspace.

The following problem remains open.

Problem. If every subspace of X has an unconditional basis is $X \sim \ell_2$?

A partial result is known. If every subspace of $(X \oplus X \oplus \cdots)_{\ell_2} = \ell_2(X)$ has an unconditional basis then $X \sim \ell_2$ [K, T-J2].

We should remark that there does exist a space (a convexified Tsirelson space) all of whose subspaces have a basis, which is not isomorphic to ℓ_2 [Jo1].

We end this section with Maurey's proof of the dichotomy theorem. This uses the ideas of Gowers in proving his block Ramsey theorem and indicates that the flavor of the argument is inherited from the Nash-Williams proof that open sets are Ramsey.

Maurey's Proof. [M3]

We may assume X has a bimonotone basis (e_i). We will say X is *H.I.* (ε), $\varepsilon > 0$, if for all block subspaces Y and Z of X there exist $y \in Y$, $z \in Z$ so that

$$\|y - z\| < \varepsilon \|y + z\| \ .$$

It is easy to check that X is H.I. iff it is H.I.(ε) for all $\varepsilon > 0$.

Claim. *If X contains no C-unconditional basic sequence then for $\varepsilon > 2/C$, X contains a block subspace which is* H.I.(ε).

Once we have established the claim a diagonal argument yields the theorem. For the rest of the argument we will work with the rational linear span of (e_i) and of block bases. Thus our subspaces will be countable sets.

For $(x,y) \in X \times X$ and a block subspace Z of X we say (x,y) *accepts* Z if for all (block) subspaces U, V of Z there exists $u \in U$, $v \in V$ with

$$(x+u, y+v) \in A \equiv \{(w,z) \in X \times X : \|w - z\| < \varepsilon \|w + z\|\} \ .$$

(x,y) *rejects* Z if for all $V \subseteq Z$ (rational block subspace), (x,y) does not accept V. Write $X \times X = \{(x_n, y_n) : n \in \mathbb{N}\}$ Inductively choose $X_1 \supseteq X_2 \supseteq \cdots$ (X_{n+1} = block subspace of X_n) so that for each n either (x_n, y_n) accepts X_n or rejects X_n. Let Z be a diagonal space of the X_n's; i.e., Z has block basis (z_i) with $(z_i)_n^\infty$ a block basis of X_n for all n. Thus for all n, (x_n, y_n) either accepts or rejects Z.

If $(0,0)$ accepts Z then Z is H.I.(ε). Assuming $(0,0)$ rejects Z we shall construct a block basis (z_k), $1 \le \|z_k\| \le 2$, so that for all $(a_i)_1^m$ and $\pm$,

$$\Big\| \sum_1^m a_i z_i \Big\| \le \frac{1}{\varepsilon} \Big\| \sum_1^m \pm a_i z_i \Big\|$$

provided that $a_k = j_k / N2^k$ for some integer $|j_k| \le N2^k$ for $k \le m$ where $N > 16/\varepsilon$ is a fixed integer. By a standard perturbation argument we get that (z_i) is $2/\varepsilon$-unconditional, which is a contradiction to our hypothesis.

In other words we shall construct (z_i) so that for $(a_i)_1^m$ as above and I, J partitioning $\{1, 2, \ldots, m\}$

$$\|x - y\| \geq \varepsilon \|x + y\| \ \text{ if } \ x = \sum_I a_k z_k \ , \ y = \sum_J a_k z_k \ ,$$

i.e., $(x, y) \notin A$ for such x and y.

Thus we wish to construct (z_i) so that for all m and all such (x, y), which we call a *reasonable pair* formed from $(z_i)_1^m$, reject Z.

Indeed if (x, y) rejects Z then for all $Z' \subseteq Z$ (x, y) does not accept Z'. Hence there exists $U, V \subseteq Z'$ so that for all $(u, v) \in U \times V$, $(x + u, y + v) \notin A$. Let $u = v = 0$ to get $\|x - y\| \geq \varepsilon \|x + y\|$.

Now if (x, y) rejects Z then for all $W \subseteq Z$ there exists $W' \subseteq W$ so that for all $w' \in W'$, $(x + w', y)$ rejects Z. Otherwise there exists $W \subseteq Z$ so that for all $U \subseteq W$ there exists $u_0 \in U$ so that $(x + u_0, y)$ accepts Z. Thus for all $V \subseteq W$ there exists $(u_1, v) \in U \times V$ so that $(x + u_0 + u_1, y + v) \in A$. Hence (x, y) accepts W which contradicts (x, y) rejects Z.

Assume $(z_i)_1^n$ are chosen so that all reasonable pairs formed from them reject Z. Since there are only finitely many such pairs there exists, by our above argument, $W \subseteq Z$ so that for all $w \in W$ and all reasonable pairs (x, y), $(x + w, y)$ rejects Z. Choose $1 < \|z_{n+1}\| < 2$, $z_{n+1} \in W$. Let (x', y') be a reasonable pair from $(z_i)_1^{n+1}$. Then $(x', y') = (x + a z_{n+1}, y)$ or $(x, y + a z_{n+1})$ where (x, y) is a reasonable pair from $(z_i)_1^n$. Both pairs reject, the second from the symmetry of A and reasonable pairs. $\square$

5. Asymptotic Notions

In this section we will explore some asymptotic notions, one of which we have already seen, namely, the spreading model of a Banach space. A second is the asymptotic structure of Maurey, Milman and Tomczak-Jaegermann [MMT] and we will mostly concentrate on these two. These structures link, in some manner, the finite dimensional and infinite dimensional structure of X.

Let's recall that every X has a 1-unconditional spreading model $(\tilde{x}_i)$ and, moreover, for some p and all n and $\varepsilon > 0$ there exists an identically distributed block basis (y_i) of (x_i) with $d_b((y_i)_1^n, (e_i)_1^n) < 1 + \varepsilon$ where (e_i) is the unit vector basis of ℓ_p (or c_0 if $p = \infty$) (Theorem 3.4).

Three natural questions are

1) Can we always find nicer spreading models, such as some c_0 or ℓ_p or perhaps either c_0, ℓ_1 or a reflexive space?
2) What does the structure of the spreading models of X tell us about X itself?
3) Is there any special structure to the set of spreading models of X?

Let's recall also that our first space not containing any c_0 or ℓ_p, T, has only ℓ_1 as a spreading model. T^* has only c_0 as a spreading model but does not contain c_0. The arbitrarily distortable S on the other hand has uncountably many spreading models (e.g., [AOST]), one of which is isomorphic to ℓ_1 [KL]. If X contains ℓ_1 (or c_0) then the James' proof that these spaces are not distortable can be used to easily show that some spreading model of X is isometric to ℓ_1 (or c_0). If X is biorthogonally distortable with a basis (e_i) then one can, given a normalized 1-unconditional 1-subsymmetric basic sequence $(w_i)_1^\infty$, for all n and $\varepsilon > 0$ renorm X so that every block basis of (e_i) admits a further block basis (z_i) with spreading model $(\tilde{z}_i)$ satisfying $d_b((\tilde{z}_i)_1^n, (w_i)_1^n) < 1 + \varepsilon$ [OS2].

T has the property that for all spreading models $(\tilde{x}_i)$, $d_b((\tilde{x}_i), (e_i)) \le 2$ where (e_i) is the unit vector basis of ℓ_1. Suppose every subspace of X has a spreading model isometric to ℓ_1. Must X contain ℓ_1? Well, no. It turns out that T has this property [OS3]. However if ℓ_1 (or c_0) is the only spreading model, isometrically, we are in business.

Theorem 5.1. [OS3] *Let (e_i) be a basis for X.*

a) *If $\|\tilde{x}_1 + \tilde{x}_2\| = 2$ for all spreading models of a normalized block basis (x_i) of (e_i) then X contains ℓ_1.*
b) *If $\|\tilde{x}_1 + \tilde{x}_2\| = 1$ for all spreading models $(\tilde{x}_i)$ of a normalized block basis (x_i) of (e_i) then X contains c_0.*
c) *If X does not contain c_0 or ℓ_1 there exists a spreading model $(\tilde{x}_i)$ of a normalized block basis (x_i) of (e_i) with $1 < \|\tilde{x}_1 + \tilde{x}_2\| < 2$.*

The proof of a) involves results of [AMT] which in turn require the definition of the Schreier classes ([AO1], [AA]). We already defined S_1; we recall this and define

$$S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \emptyset$$
$$S_1 = \{E \subseteq \mathbb{N} : |E| \le \min E\} \ .$$

Recall subsets $(E_i)_1^n$ of $\mathbb{N}$ are S_1-admissible if $n \le E_1 < \cdots < E_n$ where $E < F$ if $\max E < \min F$.

If $\alpha < \omega_1$ and S_α has been defined then

$$S_{\alpha+1} = \left\{ \bigcup_1^n E_i : (E_i)_1^n \text{ is } S_1\text{-admissible and } E_i \in S_\alpha \text{ for } i \le n \right\} .$$

If α is a limit ordinal we select $\alpha_n \uparrow \alpha$ and let $S_\alpha = \{E : \exists\, n \text{ with } E \in S_{\alpha_n} \text{ and } n \le \min E\}$. We consider $\phi < E$ or $E < \phi$ for all E so we always have $\phi \in S_\alpha$, $\alpha < \omega_1$.

The sets $(S_\alpha)_{\alpha<\omega_1}$ thus depend on the choice $\alpha_n \uparrow \alpha$ (limit ordinals) but this has no real effect on their use. First we note some of the properties of the S_α's.

Each S_α is regular as a collection of finite subsets of $\mathbb{N}$.

Definition. $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is *regular* if $\{1\} \in \mathcal{F}$ and $\mathcal{F}$ is

i) *hereditary*: $F \subseteq G \in \mathcal{F} \Rightarrow F \in \mathcal{F}$
ii) *spreading*: $F = (n_1, \dots, n_k) \in \mathcal{F}$ with $n_1 < \cdots < n_k$ and if $m_1 < \cdots < m_k$ with $n_i \le m_i$ for $i \le k$ then $(m_i)_1^k \in \mathcal{F}$.
iii) *compact*: $\mathcal{F}$ is pointwise closed in $[\mathbb{N}]^{<\omega}$ or compact in $2^{\mathbb{N}}$ where we identify F with $|_F$. Thus if $|_{F_n} \to |_F$ pointwise, $(F_n) \subseteq \mathcal{F}$, we have $F \in \mathcal{F}$.

Notation. Let $\mathcal{F}, \mathcal{G} \subseteq [\mathbb{N}]^{<\omega}$

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_1^n G_i : (G_i)_1^n \text{ is } \mathcal{F}\text{-admissible and } G_i \in \mathcal{G} \text{ for } i \le n \right\}$$

$(G_i)_1^n$ is $\mathcal{F}$-*admissible* if $G_1 < \cdots < G_n$ and $(\min G_i)_1^n \in \mathcal{F}$.

For $N = (n_i) \in [\mathbb{N}]^\omega$,

$$\mathcal{F}(N) = \{(n_i)_{i \in F} : F \in \mathcal{F}\} \quad\text{and}\quad \mathcal{F}[N] = \mathcal{F} \cap [N]^{<\omega} .$$

The Schreier classes are used to measure things that happen finitely but not infinitely, e.g., (e_i) is not equivalent to some basis (x_i) and $\mathcal{F} = \{F : (e_i)_F \overset{C}{\sim} (x_i)_F\}$ and the object becomes to put $\mathcal{F}$ or $\mathcal{F}[M]$ or $\mathcal{F}(M)$ into some S_α in some manner, and thus get a measurement of the size of $\mathcal{F}$. Here is a sample of how the classes S_α behave.

Proposition 5.2. [OTW], [AnO]

a) *Let $\alpha < \beta < \omega_1$. Then there exists n so that $n \le F \in S_\alpha \Rightarrow F \in S_\beta$.*
b) *For all $\alpha, \beta < \omega_1$ there exists $N \in [\mathbb{N}]^\omega$ so that $S_\alpha[S_\beta](N) \subseteq S_{\beta+\alpha}$.*
c) *For all $\alpha, \beta < \omega_1$ there exists $M \in [\mathbb{N}]^\omega$ so that $S_{\beta+\alpha}(M) \subseteq S_\alpha[S_\beta]$.*
d) *If $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is compact with Cantor-Bendixson index less than $\omega^\alpha + \equiv \omega^\alpha + 1$ then there exists $M \in [\mathbb{N}]^\omega$ with $\mathcal{F}(M) \subseteq S_\alpha$.*
e) *For all $M \in [\mathbb{N}]^\omega$ there exists $N \in [M]^\omega$ so that for all $\alpha < \omega_1$ if $F \in S_\alpha$, $F \subseteq N$ then $F \setminus \min(F) \in S_\alpha(M)$.*

The *Cantor-Bendixson index* of a countable compact metric space K is computed as follows:

$K^{(0)} = K$, $K^{(1)} = K' = \{k \in K : \exists\, (k_n) \subseteq K,\ k_n \to k \text{ and } k_n \ne k \text{ for all } n\}$, $K^{(\alpha+1)} = K^{(\alpha)\prime}$ and if α is a limit ordinal $K^{(\alpha)} = \bigcap_{\alpha_n < \alpha} K^{(\alpha_n)}$.

The index of $K = \inf\{\alpha : K^{(\alpha)} = \emptyset\}$.

Since K is countable, $K^{(\alpha+1)} \subsetneq K^{(\alpha)}$ if $K^{(\alpha)} \ne \emptyset$ and so the index of K is less than ω_1.

The index of S_α is $\omega^\alpha +$ since $(S_\alpha)^{(\omega^\alpha)} = \{\emptyset\}$.

Remarks. The name “Schreier” comes from the example of Schreier [Sc] of a normalized weakly null sequence (e_n) so that no subsequence admits a block basis of averages which is norm null. Schreier's space, X_1, can be defined as

$$\|(a_i)\| = \sup_{E\in S_1} \sum_{i\in E} |a_i| \ .$$

This was a precursor of T.

More generally we can define X_α using S_α instead of S_1 and get a weakly null normalized basis (e_i) so that $(e_i)_F$ is 1-equivalent to the unit vector basis of $\ell_1^{|F|}$ for all $F \in S_\alpha$.

The Schreier classes came into Banach space theory in the 1980's. The set theorists had already studied compact families of sets in $[\mathbb{N}]^{<\omega}$ in greater generality unbeknownst to us for some time (see [ATo]). It turns out that the classes (S_α) are enough, thus far, for measurements in Banach space theory, although sometimes it is useful to define the *fine Schreier sets* $(\mathcal{F}_\alpha)$, a slower evolution of the sets with $\mathcal{F}_{\omega^\alpha} = S_\alpha$.

Before returning to our preliminaries for Theorem 5.1 we mention two results, the first being from the set theorists. $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is called *thin* if for all initial segments A of $B \in \mathcal{F}$ either $A = B$ or $\mathcal{A} \notin \mathcal{F}$.

Proposition 5.3. *Let $\mathcal{F}, \mathcal{G} \subseteq [\mathbb{N}]^{<\omega}$ with $\mathcal{F}$ thin. Then there exists $L \in [\mathbb{N}]^\omega$ so that either $\mathcal{F}[L] \subseteq \mathcal{G}$ or $\mathcal{F}[L] \cap \mathcal{G} = \emptyset$.*

Proof. We first show there exists $L_0 \in [\mathbb{N}]^\omega$ such that either $\mathcal{F} \cap [L_0]^{<\omega} = \emptyset$ or for all $M \in [L_0]^\omega$ some initial segment of M is in $\mathcal{F}$. Let

$$\mathcal{A} = \{L : \text{ some initial segment of } L \text{ is in } \ \mathcal{F}\}$$

$\mathcal{A}$ is Ramsey so there exists $L_0 \in [\mathbb{N}]^\omega$ with either $[L_0] \subseteq \mathcal{A}$ or $[L_0] \cap \mathcal{A} = \emptyset$. In the second case $\mathcal{F} \cap [L_0]^{<\omega} = \emptyset$, so we may assume for all $M \in [\mathbb{N}]$, some initial segment of M is in $\mathcal{F}$.

For $L = (\ell_i)_1^\infty \in [\mathbb{N}]^\omega$ we define $f(L) = n$ if $(\ell_i)_1^n \in \mathcal{F}$. Set $G(L) = \{n : (\ell_1, \ldots, \ell_n) \in \mathcal{G}\}$. $G(L)$ could be empty, finite or infinite. Let $\beta = \{L \in [\mathbb{N}]^\omega : f(L) \in G(L)\}$. β is Ramsey so there exists $L \in [\mathbb{N}]^\omega$ with $[L]^\omega \subseteq \beta$ or $[L]^\omega \cap \beta = \emptyset$. If $[L]^\omega \subseteq \beta$ then for all $F \in \mathcal{F}[L]$, F is an initial segment of some $M \in [L]^\omega$ so $f(M) \in G(M) \Rightarrow F \in \mathcal{G}$. If $[L]^\omega \cap \beta = \emptyset$ then for all $F \in \mathcal{F}[L]$, if F is an initial segment of $M \subseteq L$ then $f(M) \notin G(M)$ so $F \notin \mathcal{G}$. □

We have the following (see also [Ju], [Ga]),

Corollary 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be regular families in $[\mathbb{N}]^{<\omega}$. Then there exists $M \in [\mathbb{N}]^\omega$ with $\mathcal{F}[M] \subseteq \mathcal{G}$ or $\mathcal{G}[M] \subseteq \mathcal{F}$.*

Proof. We let $\mathcal{F}_{\max} = \{F \in \mathcal{F} : F$ is not a proper initial segment of an element in $\mathcal{F}\}$. By 5.3 we obtain M with $\mathcal{F}_{\max}[M] \subseteq \mathcal{G}$ or $\mathcal{F}_{\max}[M] \cap \mathcal{G} = \emptyset$. If $F \in \mathcal{F}[M]$ there exists $F_1 \in \mathcal{F}_{\max}[M]$ so that F is an initial segment of F_1. The result follows. □

Our second result connects us with partial unconditionality. One result we did not mention earlier was an unpublished result of Rosenthal which motivated Elton's theorem: If a sequence (1_{E_n}) of indicator functions is weakly null in $C(K)$ then some subsequence is unconditional.

Theorem 5.4. [GOW] *Let $(f_n) \subseteq C(K)$ be normalized weakly null with for some $\delta > 0$, all n, $f_n(k) \neq 0 \Rightarrow |f_n(k)| \geq \delta$. Then there exist a subsequence (f_{n_i}) of (f_n) and a subsequence (e_{m_i}) of the unit vector basis for some Schreier space X_α so that $(f_{n_i}) \sim (e_{m_i})$.*

We return now to the discussion of results used to prove Theorem 5.1 a). Following [AMT] we define the *α-averages over M*, $(\alpha_n^M)_{n=1}^\infty \subseteq S_{\ell_1}^+ \cap c_{00}$. Roughly speaking we inductively average as long as we can so that the supports lie in S_α. For example if $M = (2,7,10,11,\dots,19,37,\dots)$, then (for $\alpha = 1$)

$$1_1^M = \frac{1}{2}(e_2+e_7)\ , \quad 1_2^M = \frac{1}{10}\sum_{10}^{19} e_i\ , \quad 1_3^M = \frac{1}{37}(e_{37}+\cdots),\dots\ .$$

Then (for $\alpha = 2$)

$$2_1^M = \frac{1}{2}(1_1^M + 1_2^M)\ , \quad 2_2^M = \frac{1}{37}(1_3^M+\cdots+1_{39}^M),\dots\ .$$

Let $M = (m_i)$ and set $O_n^M = e_{m_n}$. Assume (α_n^M) have been defined. Set

$$(\alpha+1)_1^M = \frac{1}{m_1}\sum_{i=1}^{m_1}\alpha_i^M$$

and if $(\alpha+1)_j^M$ has been defined for $j \le n$ let

$$M_{n+1} = \{m_i : m_i > \max\operatorname{supp}(\alpha+1)_n^M\} \quad \text{and} \quad (\alpha+1)_{n+1}^M = (\alpha+1)_1^{M_{n+1}}\ .$$

If $\alpha_k \uparrow \alpha$, a limit ordinal (used to define S_α) we set
$\alpha_1^M = (\alpha_{m_1})_1^M$ and for $n > 1$
$\alpha_n^M = (\alpha_{k_n})_1^{M_n}$ where $M_n = \{m \in M : m > \operatorname{supp}\alpha_{n-1}^M\}$ and $k_n = \min M_n$.

Proposition 5.5. [AMT]

1) *For $\alpha < \omega_1$, $(\alpha_n^M)_{n=1}^\infty \equiv (\alpha_n^M(e_i))_{n=1}^\infty$ is a convex block basis of (e_i) and $\cup_k \operatorname{supp}(\alpha_k^M) = M$. Moreover $\operatorname{supp}\alpha_n^M \in (S_\alpha)_{\max}$ for all n.*
2) *If $M \in [\mathbb{N}]^\omega$, $\alpha < \omega_1$ and $(n_k) \in [\mathbb{N}]$ then $\alpha_{n_k}^M = \alpha_k^{M'}$ where $M' = \cup_k \operatorname{supp}(\alpha_{n_k}^M)$*

α-averages lead to a quantified version of a theorem of Ptak [P] and of Mazur's theorem.

Theorem 5.6. [P] *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ be hereditary. Let $\varepsilon > 0$. Suppose for all $(a_i) \subseteq S_{\ell_1}^+$ there exits $F \in \mathcal{F}$ with $\sum_F a_i \ge \varepsilon$. Then there exists $M \in [\mathbb{N}]^\omega$ with $[M]^{<\omega} \subseteq \mathcal{F}$.*

Proof. Let $X = \overline{(c_{00}, \|\cdot\|_{\mathcal{F}})}$. Then the unit vector basis for X is equivalent to the ℓ_1 basis and $X \subseteq C(\overline{\mathcal{F}})$. If $\overline{\mathcal{F}}$ is countable we get a contradiction since $\ell_1 \not\hookrightarrow C(K)$, K any countable compact metric space, so $\overline{\mathcal{F}}$ is uncountable. In particular some infinite $M \in \overline{\mathcal{F}}$ and the result follows. □

For $x \in \sum a_i e_i \in c_{00}$ and $F \subseteq \mathbb{N}$ we define $\langle x, F\rangle = \sum_F a_i$.

Definition. [AMT] Let $\mathcal{F}$ be an hereditary collection of subsets of $\mathbb{N}$, $M \in [\mathbb{N}]^\omega$ and $\varepsilon > 0$, $\alpha < \omega_1$. $\mathcal{F}$ is *(M,α,ε)-large* if for all $N \in [M]^\omega$, $n \in \mathbb{N}$

$$\sup_{F\in\mathcal{F}}\langle \alpha_n^N, F\rangle \ge \varepsilon\ .$$

Theorem 5.7. [AMT] *If $\mathcal{F}$ is (M,α,ε) large then there exists $N \in [M]^\omega$ with $\mathcal{F} \supseteq S_\alpha(N)$.*

Mazur's theorem yields that if (e_i) is normalized weakly null then some convex block basis is norm null.

Theorem 5.8. [AMT] *Let (e_i) be normalized weakly null. Then there exists $\alpha < \omega_1$ so that for all $M \in [\mathbb{N}]^\omega$, $\lim_n \|\alpha_n^M(e_i)\| = 0$.*

We sketch now how these ingredients are used to prove Theorem 5.1 a). Recall we have a normalized basis (e_i) for X so that if $(\tilde{x}_i)$ is any spreading model of a normalized block basis of (e_i) then $\|\tilde{x}_1 + \tilde{x}_2\| = 2$. By Rosenthal's ℓ_1 theorem it suffices to show (e_i) is not weakly null. Indeed the hypothesis on (e_i) passes to its block bases so no (x_i) could be weakly null and thus $\ell_1 \hookrightarrow X$.

To do this one shows by induction on $\alpha < \omega_1$ that if

$$\mathcal{F}(\varepsilon) = \{F \subseteq \mathbb{N} : \text{ there exists } \ x^* \in S_{X^*} \ \text{ with } \ x^*(e_i) \geq 1 - \varepsilon \ \text{ for } \ i \in F\}$$

then

(P_α) For all $M \in [\mathbb{N}]^\omega$ and $\varepsilon > 0$ there exists $N \in [M]^\omega$ with $\mathcal{F}(\varepsilon) \supseteq S_\alpha(N)$.

It then follows that $\mathcal{F}(\varepsilon) \supseteq [M]^{<\omega}$ for some $M \in [\mathbb{N}]^\omega$ since its index exceeds ω^α for all $\alpha < \omega_1$. This is done by using Ramsey theory, Theorem 5.7, some analytical arguments and e) in 5.2. □

As we have noted before, ℓ_1 and c_0 are rather special among the ℓ_p spaces. The following problem remains open.

Problem. Let $1 < p < \infty$ and let (e_i) be a basis for X with the property that all spreading models $(\tilde{x}_i)$ of normalized block bases are 1-equivalent to the unit vector basis of ℓ_p. Does $\ell_p \hookrightarrow X$?

From Theorem 5.1 a), if all spreading models of X (of block bases) are isometric to ℓ_1 we obtain $\ell_1 \hookrightarrow X$, and as noted for all $X \subseteq T$, ℓ_1 is an isometric spreading model of some sequence in X. It turns out that if ℓ_1 is an isometric spreading model of $(X, |\cdot|)$ under all equivalent norms $|\cdot|$ on X then X contains ℓ_1.

Theorem 5.9. [OS4] *The following are equivalent for a space X.*

1) *X contains an isomorph of ℓ_1 (respectively, c_0).*
2) *For all equivalent norms $|\cdot|$ on X there exists a $|\cdot|$-normalized basic sequence (x_i) with spreading model $(\tilde{x}_i)$ 1-equivalent to the unit vector basis of ℓ_1 (respectively, c_0).*
3) *For all equivalent norms on X there exists a normalized (respectively, and weakly null) basic sequence (x_i) having spreading model $(\tilde{x}_i)$ satisfying $\|\tilde{x}_1 \pm \tilde{x}_2\| = 2$ (respectively, $\|\tilde{x}_1 + \tilde{x}_2\| = 1$).*

It turns out that one can deduce this theorem from one particular norm. First we start out with $(X, \|\cdot\|)$ being *strictly convex* ($\|x\| = \|y\| = 1$ and $\|x+y\| = 2 \Rightarrow x = y$). Every X has such an equivalent norm. We now choose a countable dense set $C \subseteq X$ which is closed under rational linear combinations. Then choose $(\rho_c)_{c \in C} \subseteq (0, \infty)$ so that $\sum_{c \in C} \rho_c (1 + \|c\|) < \infty$ and set

$$|x| = \sum_{c \in C} \rho_c \|x\|_c \ .$$

Recall, $\|x\|_c = \big\|c\|x\| + x\big\| \ + \ \big\|c\|x\| - x\big\|$ is an equivalent norm on X for each c. The idea of the proof is roughly this. Assume (x_n) is normalized in $(X, \|\cdot\|)$ and admits a spreading model over X for $|\cdot|$ and every $|\cdot|_c$, $c \in C$. Suppose $|\tilde{x}_1 + \tilde{x}_2| = 2|\tilde{x}_1|$. We obtain that $\|\tilde{x}_1 + \tilde{x}_2\|_c = 2\|\tilde{x}_1\|_c$ for all $c \in C$. This enables us to deduce, after some work, things like

$$\lim_m \lim_n \|y + \beta_1 x_m + \beta_2 x_n\| = \lim_m \|y + (\beta_1 + \beta_2) x_m\|$$

if $\beta_1, \beta_2 \geq 0$, and ultimately get ℓ_1 in X.

The argument above was actually constructed to solve a problem of Milman of which the theorem was a by-product.

Theorem 5.10. [OS4] *X is reflexive (if and) only if there exists an equivalent norm $|\cdot|$ on X satisfying for any bounded $(x_n) \subseteq X$:*

$$\text{If} \quad \lim_m \lim_n |x_m + x_n| = 2 \lim_n |x_n| \quad \text{then } (x_n) \text{ is norm convergent.}$$

The condition lies somewhere between strictly convex and uniformly convex. The if part follows from James' characterization [J4] of reflexivity: Let $|x^*| = 1$ and choose $(x_n) \subseteq S_{X,|\cdot|}$ with $x^*(x_n) \to 1$. Then $\lim_m \lim_n |x_m + x_n| = 2$ so $x_m \to x$ with $x^*(x) = 1$. More generally we have

Theorem 5.11. *Every X admits a strictly convex norm $|\cdot|$ satisfying: if $(x_m) \subseteq X$ is relatively weakly compact and if*

$$\lim_m \lim_n |x_m + x_n| = 2 \lim_n |x_n|$$

then (x_n) is norm convergent.

As for the question if c_0 or some ℓ_p must always be a spreading model of X the answer is no. An example [OS5] is given by $X = \overline{(c_{00}, \|\cdot\|)}$ where $\|\cdot\|$ satisfies the implicit equation below. As before $f(n) = \log_2(n+1)$ and we choose $(n_k) \in [\mathbb{N}]$ with

$$\sum_{k=1}^{\infty} \frac{1}{f(n_k)} < \frac{1}{10}$$

$$\|x\| = \|x\|_\infty \vee \left(\sum_{k=1}^{\infty} \|x\|_{n_k}^2 \right)^{1/2} \quad \text{where}$$

$$\|x\|_k = \max \left\{ \frac{1}{f(k)} \sum_{i=1}^{k} \|E_i x\| : E_1 < \cdots < E_k \right\} .$$

(e_i) is an unconditional subsymmetric basis for the reflexive space X. One need only check that no spreading model $(\tilde{x}_i)$ of a normalized block basis is equivalent to the unit vector basis of c_0 or any ℓ_p. Since $\| \sum_1^{n_k} \tilde{x}_i \| \geq n_k / f(n_k)$ the only possibility is ℓ_1. Now if $(\tilde{x}_i)$ is an ℓ_1 basis by replacing it by an identically distributed block basis (James' proof that ℓ_1 is not distortable) we could assume

$$\left\| \sum a_i \tilde{x}_i \right\| \geq .99 \sum |a_i| \ . \tag{5.1}$$

Next we consider $d_i = (d_{i,k})_{k=1}^\infty \in S_{\ell_2}$ given by $d_{i,k} = \|x_i\|_{n_k}$, $1 = \|x_i\| = (\sum_k \|d_{i,k}\|_2^2)^{1/2}$. Passing to a subsequence $d_i \xrightarrow{w} d \in B_{\ell_2}$ and from (5.1) we can argue $\|d\|_2 \geq .99$. Since d is mostly supported in some finite interval we get that the ℓ_1 nature of spreading model $(\tilde{x}_i)$ must come from (essentially) only part of the norm: there exists k_0 so that for all N if n is large and $n \leq i_1 < \cdots < i_N$, then

$$N \approx \left(\sum_{k=1}^{k_0} \left\| \sum_{j=1}^{N} x_{i_j} \right\|_{n_k}^2 \right)^{1/2} .$$

One shows, after some calculations, that this cannot be true for large enough N.

It turns out that the proof carries over to yield $\ell_1 \not\hookrightarrow [(\tilde{x}_i)]$ for any spreading model.

One can do more. There exists a space X so that no spreading model is isomorphic to ℓ_1, c_0 or a reflexive space [AOST]. This is accomplished by constructing X so that every spreading model contains ℓ_1 but none is isomorphic to ℓ_1. X is reflexive with an unconditional basis.

Let $SP(X)$ denote the set of all spreading models of X generated by normalized basic sequences and $SP_w(X)$ denote those generated by normalized weakly null basic sequences in X. Of course if X is reflexive $SP(X) = SP_w(X)$. As we have seen we need not have any ℓ_p or c_0 in $SP(X)$ so we cannot strengthen the earlier consequence of Krivine's theorem: For all $(\tilde{x}_i)$ and some p there exists for $n \in \mathbb{N}$ and $\varepsilon > 0$ an identically distributed block basis (y_i) of (x_i) with $d_b((\tilde{y}_i)_1^n, (e_i)_1^n) < 1 + \varepsilon$ where (e_i) is the unit vector basis of ℓ_p (c_0 if $p = \infty$). Also spreading models are not transitive: if $(\tilde{x}_i) \in SP(X)$ and $(\tilde{y}_i) \in SP[(\tilde{x}_i)]$ then $(\tilde{y}_i)$ need not be in $SP(X)$.

This problem remains open.

Problem. For all X does there exist n and spaces $X_0 = X, X_1, \dots, X_n$ so that $X_n \sim \ell_p$ for some $1 \le p < \infty$ or c_0 and $X_{i+1} \in SP(X_i)$ for $i < n$?

The structure of $SP_w(X)$ was investigated in [AOST] in terms of a partial order: $(\tilde{x}_i) \ge (\tilde{y}_i)$ if $(\tilde{x}_i)$ *dominates* $(\tilde{y}_i)$, i.e., for some $C < \infty$ and all $(a_i) \subseteq \mathbb{R}$,

$$\Big\| \sum a_i \tilde{y}_i \Big\| \le C \Big\| \sum a_i \tilde{x}_i \Big\| .$$

Of course we identify equivalent spreading models.

Theorem 5.12. [AOST] $(SP_w(X), \le)$ *has the following properties*

a) *It is a* semi-lattice, *i.e., for all $(\tilde{x}_i), (\tilde{y}_i) \in SP_w(X)$ the least upper bound $(\tilde{x}_i) \vee (\tilde{y}_i) \in SP_w(X)$. That is there exists $(\tilde{z}_i) \in SP_w(X)$ with $\| \sum a_i \tilde{z}_i \| \sim \| \sum a_i \tilde{x}_i \| \vee \| \sum a_i \tilde{y}_i \|$.*

b) *If $A \subseteq SP_w(X)$ is countable then A has an upper bound in $SP_w(X)$. Moreover if $(\tilde{x}_i^{(n)})_{i=1}^\infty \in SP_w(X)$ for $n \in \mathbb{N}$ and $\sum_{n=1}^\infty C_n^{-1} < \infty$, $C_n > 0$, there exists $(\tilde{y}_i) \in SP_w(X)$ and $K \le \sum C_n^{-1}$ so that $(\tilde{y}_i)$ KC_n-dominates $(\tilde{x}_i^{(n)})_{i=1}^\infty$ for each n. If each $(\tilde{x}_i^{(n)})$ is not equivalent to the unit vector basis of ℓ_1 then neither is $(\tilde{y}_i)$.*

a) is a special case of the proof of b). The idea is to first use Ramsey theory for a fixed n and $(x_i^{(1)}), \dots, (x_i^{(n)})$ to select for a given $\varepsilon > 0$, $L \in [\mathbb{N}]$ so that if we have integers in L,

$$k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_2^{(n)} < \cdots < k_n^{(n)}$$
$$\text{and} \quad \ell_1^{(1)} < \ell_1^{(2)} < \cdots < \ell_1^{(n)} < \ell_2^{(1)} < \cdots < \ell_2^{(n)} < \cdots < \ell_n^{(n)}$$

and $(a_i^{(j)})_{i,j=1}^n \subseteq [-1, 1]$ then

$$\left| \Big\| \sum_{i=1}^n \sum_{j=1}^n a_i^{(j)} x_{\ell_i^{(j)}}^{(j)} \Big\| - \Big\| \sum_{i=1}^n \sum_{j=1}^n a_i^{(j)} x_{k_i^{(j)}}^{(j)} \Big\| \right| < \varepsilon .$$

Moreover each $(x_{k_1^{(1)}}^{(1)}, x_{k_1^{(2)}}^{(2)}, \cdots, x_{k_1^{(n)}}^{(n)}, \cdots, x_{k_n^{(n)}}^{(n)})$ is suppression $1 + \varepsilon$-unconditional.

Then one uses a diagonal argument for $n \to \infty$ and $\varepsilon_n \to 0$ and, after carefully relabeling appropriate subsequences, sets $y_j = \sum_{i=1}^j C_i^{-1} x_i^{(i)}$, and then normalizes.

So what else can be said about $SP_w(X)$? The space X of [AOST] with no spreading model isomorphic to c_0, ℓ_1 or reflexive has $SP_w(Y)$ uncountable for all $Y \subseteq X$. Also if $SP_w(X)$ admits an infinite increasing sequence then it admits an uncountable strictly increasing chain [Sa1]. If $SP_w(X)$ is uncountable then it admits an antichain the size of the continuum [Do].

Results on possible realizations of various semi-lattices as $SP_w(X)$ are given in [DOS]. For example it is proved that if L is a countable lattice with a minimum element and L does not admit an infinite increasing sequence then there exists a reflexive X_L with $SP_w(X_L)$ order isomorphic to L. If L is any countable semi-lattice not admitting an infinite increasing chain then there exists X with $SP_w(X)$ order isomorphic to L [LeT6].

If $SP_w(X)$ is countable then by a diagonal argument one can find $Y \subseteq X$ so that $SP_w(Y) = SP_w(Z)$ for all $Z \subseteq Y$. Presumably this is not true in general which is a weakness of spreading model theory versus asymptotic structure, discussed later.

Problem. a) If $|SP_w(X)| = 1$ must some ℓ_p or c_0 be a spreading model?
b) If $SP_w(X)$ is countable must some ℓ_p or c_0 be a spreading model?

Partial results to a) are known [AOST], [Sa1], [DOS]. a) is due to S. Argyros.

One can get a slightly broader swath of vectors yielding the same spreading model. Given finite dimensional spaces $G_n \subseteq X$ with $\dim G_n \to \infty$, one can find $k_n \uparrow \infty$, $F_n \subseteq G_{k_n}$, $\dim F_n = n$ and for all $x_n, y_n \in S_{F_n}$, $(\tilde{x}_n) \overset{1}{\simeq} (\tilde{y}_n)$, and even over X. In particular this unique spreading model is 1-unconditional over X [ORS].

An extension of spreading models to *asymptotic models* is studied in [HO]. These are generated by a sequence of normalized bases instead of just one. The theory often carries over and in this setting the unique asymptotic model problem has an affirmative answer.

Spreading models exist by virtue of the fact that given a basic sequence in S_X one can nearly stabilize $\|\sum_F a_i x_i\|$ over $F \in S_1$. It is natural to ask if we could obtain similar stabilizations over S_2-sets (or even S_α-sets) but this does not work, for example in T. However a number of results have been obtained about higher order ℓ_1 (or ℓ_p) — spreading models (see e.g. [LeT1, LeT2, LeT3], [JuO], [ADM]). A normalized basic sequence (x_i) is said to be a K-ℓ_1-S_α spreading model if for all $F \in S_\alpha$, $(x_i)_F$ is K-equivalent to the unit vector basis of $\ell_1^{|F|}$.

For example the unit vector basis of T is a 2^{-n}-ℓ_1-S_n-spreading model for all n but T contains no ℓ_1-S_ω-spreading model.

We conclude the spreading model part of this section with a theorem that uses spreading models in one section of its proof. We let $D(X)$ be the diameter under the Banach-Mazur distance of $\{Y : Y \sim X\}$.

Theorem 5.13. [JO1] *$D(X) = \infty$ for all X.*

We will outline the steps in the proof. This will also give us the opportunity to define Bourgain's ordinal index.

Remark. It is known that for some $c \in (0,1)$, if $\dim F = n$ then $D(F) \geq c\,n$ [Gl]. Thus it seems certain that $D(X) = \infty$ but this does not seem to follow from the finite dimensional result.

Outline of proof of Theorem 5.13.

1) [Pe1] If $Y \subseteq X$ and $|\cdot|$ is an equivalent norm on Y then $|\cdot|$ extends to an equivalent norm on X.

2) [LP] If X has basis (x_i) then for all $n \in \mathbb{N}$ there exists an equivalent norm $|\cdot|_n$ on X so that under $|\cdot|$, (x_i) is *block n-unconditional with constant* 2. This means that for all

block bases $(y_i)_1^n$ of (x_i)

$$\Big\| \sum_1^n \pm y_i \Big\| \le 2 \Big\| \sum_1^n y_i \Big\| \ .$$

By renorming $C[0,1] \supseteq X$ using 2) we have

3) For all n there exists an equivalent norm $|\cdot|_n$ on X so that for all $\varepsilon > 0$ every normalized weakly null sequence in $(X, |\cdot|_n)$ admits a subsequence which is block n-unconditional with constant $2+\varepsilon$.

The next step is proved by an adaptation of the Maurey-Rosenthal construction.

4) Let (x_n) be normalized weakly null with spreading model $(\tilde{x}_n)$ not equivalent to either the unit vector basis of ℓ_1 or c_0. Then for all $C < \infty$ there exist $n \in \mathbb{N}$, a subsequence $(y_i) \subseteq (x_i)$ and an equivalent norm $|\cdot|$ on $[(y_i)]$ so that no subsequence of (y_i) is block n-unconditional with constant C, for $|\cdot|$.

So if we can show that $D(X) < \infty$ implies X contains an element in $SP_w(X)$ which is not equivalent to the unit vector basis of either ℓ_1 or c_0 then we will be done. Indeed, now by 1) and 4) given C and n we can renorm X to $(X, |\cdot|)$ so that it contains a normalized weakly null sequence not admitting a block n-unconditional subsequence with constant C. On the other hand if $D(X) < \infty$ then by 3) for some K and all n every weakly null sequence in $S_{(X,|\cdot|)}$ admits a block n-unconditional subsequence with constant K which is a contradiction.

We note that the condition $(\tilde{x}_i)$ is not ℓ_1 nor c_0 can be characterized by $\lim_m \| \sum_1^m \tilde{x}_i \| / m = 0$ and $\lim_m \| \sum_1^m \tilde{x}_i \| = \infty$, and this is used in the proof of 4) and 5).

5) [AOST] Let $(\tilde{x}_i^{(n)})_{i=1}^\infty \in SP_w(X)$ for all n. Assume each $(\tilde{x}_i^{(n)})$ is not equivalent to the unit vector basis of ℓ_1. Then there exists $(\tilde{y}_i) \in SP_w(X)$, not equivalent to the unit vector basis of ℓ_1, so that for some $\lambda > 0$ $(\tilde{y}_i)$ $\lambda 2^{-n}$-dominates $(\tilde{x}_i^{(n)})$,

$$\Big\| \sum a_i \tilde{y}_i \Big\| \ge \lambda 2^{-n} \Big\| \sum a_i \tilde{x}_i^{(n)} \Big\| \ , \quad \text{for all} \quad n \in \mathbb{N} \ .$$

This follows from Theorem 5.12.

Definition. X is K-*elastic* if for all Y, $Y \hookrightarrow X \Rightarrow Y \overset{K}{\hookrightarrow} X$.

Note that $C[0,1]$ is 1-elastic. Also if $D(X) < \infty$ then X is $D(X)$-elastic, using 1).

6) If X is elastic then $c_0 \hookrightarrow X$.

We will say more about the proof of 6) later. Now we complete the proof. Assume $D(X) < \infty$. Using that $c_0 \hookrightarrow X$, and renorming c_0 by

$$|(a_i)|_n = \sup \Big\{ \Big| \sum_F a_i \Big| : |F| = k_n \Big\}$$

where $2^{-n} k_n \to \infty$ we can find, using X is elastic, $(\tilde{x}_i^{(n)}) \in SP_w(X)$, equivalent to the c_0-basis, satisfying for some K and all n,

$$\Big\| \sum a_i \tilde{x}_i^{(n)} \Big\| \ge K^{-1} |(a_i)|_n \ .$$

By 5) we find $(\tilde{x}_i) \in SP_w(X)$ satisfying for all n,

$$\Big\| \sum_1^{k_n} \tilde{x}_i \Big\| \ge \lambda K^{-1} 2^{-n} k_n \to \infty \ .$$

Thus $(\tilde{x}_i)$ is not equivalent to the unit vector basis of ℓ_1 nor c_0.

Before discussing 6) we need to introduce an ordinal index of Bourgain [Bo] (see also [AGR]). Fix K and consider $T(K) = \{(x_i)_1^n \subseteq S_X\colon (x_i)_1^n$ is K-equivalent to the unit vector basis of $\ell_\infty^n\}$. $T(K)$ is a tree ordered by extension: $(x_i)_1^m \le (y_i)_1^n$ if $m \le n$ and $x_i = y_i$ for $i \le m$. If $T(K)$ is *well founded* (there are no infinite branches, i.e., there does not exist $(x_i)_1^\infty \subseteq S_X$ so that for all n, $(x_i)_1^n \in T(K)$) then we can define its index as follows.

$$T(K)' = \{(x_i)_1^n \in T(K) : \text{ there exists } x_{n+1} \text{ with } (x_i)_1^{n+1} \in T(K)\} .$$

We then define $T_0 = T(K)$, $T_{\alpha+1} = T'_\alpha$ and if α is a limit ordinal $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$. Then $o(T(K)) = \inf\{\alpha : T_\alpha = \emptyset\}$. Now $T(K)$ is a *closed tree*: if $(x_i^{(n)})_{i=1}^m \in T(K)$ for $n \in \mathbb{N}$ and $x_i^{(n)} \xrightarrow{n\to\infty} x_i$ then $(x_i)_1^m \in T(K)$.

It follows, as Bourgain observed, that if $T(K)$ is well founded then $o(T(K)) < \omega_1$. The c_0-index of X is defined as $\sup\{o(T(K)) : K < \infty\}$. Hence we have that $c_0 \not\hookrightarrow X$ iff the c_0-index of X is less than ω_1.

Thus to prove 6) X elastic $\Rightarrow c_0 \hookrightarrow X$, it suffices to show that for all $\beta < \omega_1$, $o(T(K)) \ge \beta$ when X is K-elastic. The proof is rather involved so we shall not give it here. The rough idea is to build c_0-trees of increasing complexity and use the elastic property to control the c_0-constant. □

Remarks. Bourgain's index can be used also for ℓ_1 or ℓ_p or any unconditional basis or even any basis, although in the latter case one needs to allow seminormalized trees.

As we mentioned $C[0,1]$ is 1-elastic. It remains open if every elastic space must contain $C[0,1]$. Partial results appear in [JO1].

Problem. If X is elastic does $C[0,1] \hookrightarrow X$?

Also there is a very pretty result of Nicole Tomczak-Jaegermann [T-J2] which uses index theory. One can define the unconditional index of X via the trees $T(K) = \{(x_i)_1^n \subseteq S_X :$ $(x_i)_1^n$ is K-unconditional$\}$ and obtain that X contains an unconditional basic sequence iff for some K and all $\beta < \omega_1$, $o(T(K)) \ge \beta$.

As we mentioned it is not known if spaces X of bounded distortion exist (X is distortable but not arbitrarily distortable). But if they do exist we can say something about their structure. In [MT] it is shown using Krivine's theorem that such an X contains an asymptotic ℓ_p (or c_0) basis.

Definition. A basis (e_i) is K-*asymptotic* ℓ_p if for all n and normalized block bases $(x_i)_1^n$ of $(e_i)_{i=n}^\infty$, $(x_i)_1^n$ is K-equivalent to the unit vector basis of ℓ_p^n (we call this K-*asymptotic* c_0 if $p = \infty$).

The unit vector basis of T is 2-asymptotic ℓ_1.

Theorem 5.14. [T-J2] *If X has bounded distortion then X contains an unconditional basic sequence.*

Remarks. The proof does not require X to be distortable. Maurey [M1] went on to show that a space X with an asymptotic ℓ_p unconditional basis is biorthogonally distortable, if ℓ_1 is not finitely representable in X. Thus again we see that T is a prime candidate for a space of bounded distortion.

Sketch of proof. Assume X has an asymptotic ℓ_p basis and is of D-bounded distortion. One shows that for $D' > D$ and each $\alpha < \omega_1$ there is a block basis (x_i^α) of (e_i) which is

$4D'$-S_α-unconditional, i.e., $(x_i^\alpha)_{i\in F}$ is $4D'$-unconditional for each $F\in S_\alpha$. This makes the $4D'$-unconditional index, $o(T(4D'))\geq\omega^\alpha$ for all α and we are then done.

To do this we find (x_i^α) so that if $(E_i)_1^n$ is S_α-admissible (i.e., $E_1<\cdots<E_n$ and $(\min E_i)_1^n\in S_\alpha$) then $\|\sum_1^n \pm E_i x\|\leq 4D'\|x\|$ for all $x\in\langle (x_i^\alpha)_i\rangle$. This is done by induction on α.

To get $(x_i^{\alpha+1})$ we define on $\langle (x_i^\alpha)\rangle$

$$|x|=\sup\left\{\Big\|\sum_1^n \pm E_i x\Big\| : (E_i)_1^n \text{ is } S_{\alpha+1}\text{-admissible}\right\}.$$

We can write $|x|=\|\sum_{j=1}^m\sum_{i\in A_j}\pm E_i x\|$ where the vectors $(\sum_{i\in A_j}\pm E_i x)_{j=1}^m$ are S_1-admissible, and each $(E_i)_{i\in A_j}$ is S_α-admissible. Thus, using the C-asymptotic ℓ_p property,

$$\|x\|\leq |x|\overset{C}{\sim}\left(\sum_{j=1}^m\Big\|\sum_{i\in A_j}\pm E_i x\Big\|^p\right)^{1/p}\leq\left(\sum_{j=1}^m K\Big\|\sum_{i\in A_j}E_i x\Big\|^p\right)^{1/p}\leq C\,K\|x\|$$

where K is the S_α-unconditional constant of $(x_i^\alpha)_i$.

Thus $|\cdot|$ is an equivalent norm on $[(x_i^\alpha)]$. One then uses a combinatorial argument to show if $(E_i)_1^n$ is $S_{\alpha+1}$-admissible then $|\sum_1^n\pm E_i x|\leq 4|x|$. Since X is of D-bounded distortion there exists a block basis $(x_i^{\alpha+1})$ of (x_i^α) so that

$$\Big\|\sum_1^n\pm E_i x\Big\|\leq 4D'\|x\| \quad\text{for}\quad x\in\langle x_i^{\alpha+1}\rangle$$

with $(E_i)_1^n$ $S_{\alpha+1}$-admissible. The limit ordinal case is easy. □

In light of these results it is natural to ask if a space X with an asymptotic ℓ_1 basis can be arbitrarily distortable. Argyros and Deliyanni [AD] showed this to be the case by constructing two examples, the first of which has an unconditional basis and they called it a *mixed Tsirelson* space, and the second (which we do not present) is H.I.

We can regard $T=T(\frac{1}{2},S_1)$. More generally given $0<\theta<1$ and $\alpha<\omega_1$ we can define $T(\theta,S_\alpha)$ via the implicit equation

$$\|x\|=\|x\|_\infty\vee\sup\left\{\theta\sum_{i=1}^n\|E_i x\| : (E_i)_1^n \text{ is } S_\alpha\text{-admissible}\right\}.$$

These spaces are much like T: reflexive, (e_i) is a 1-unconditional asymptotic ℓ_1 basis for $T(\theta,S_\alpha)$ which is $\frac{1}{\theta}-\varepsilon$ distortable for all $\varepsilon>0$. Like T it is not known if these are arbitrarily distortable. The mixed Tsirelson spaces combine these norms.

Definition. Let $\alpha_1<\alpha_2<\cdots$ be ordinals and let $\theta_n\downarrow 0$, $\theta_n<1$ for all n. $T(\theta_n,S_{\alpha_n})_{n\in\mathbb{N}}$ is the completion of c_{00} under the implicit norm

$$\|x\|=\|x\|_\infty\vee\sup_{m\in\mathbb{N}}\sup\left\{\theta_m\sum_{i=1}^m\|E_i x\| : (E_i)_1^n \text{ is } S_{\alpha_m}\text{-admissible}\right\}.$$

As with T, $T(\theta_n,S_{\alpha_n})_{n\in\mathbb{N}}$ has a 1-unconditional basis, is reflexive and is asymptotic ℓ_1 [AD]. Moreover in certain circumstances it is arbitrarily distortable. Perhaps the easiest case is for $T(\frac{1}{n+1},S_n)_{n\in\mathbb{N}}$ [AnO]. More generally $T(\theta_n,S_n)_{n\in\mathbb{N}}$ is arbitrarily distortable if $\frac{\theta_n}{\theta^n}\to 0$ where $\theta=\lim_n\theta_n^{1/n}$.

A number of papers have studied mixed Tsirelson, higher order Tsirelson, and asymptotic ℓ_1 spaces in terms of various indices [OTW] or higher order S_α-ℓ_1 spreading models and other properties (e.g., [LeT4, LeT5]).

We have defined what it means for a basis (x_i) for X to be asymptotic ℓ_p. This is a coordinatized notion, which can be generalized in several ways [MMT]. Let $(e_i)_1^n$ be a normalized basis. We will say

Definition.

$(e_i)_1^n \in \{X\}_n^t$, *the tail asymptotic structure of* X *w.r.t.* (x_i), *if*

$\forall\ \varepsilon > 0\ \forall\ k_1\ \exists\ y_1 \in S_{\langle x_i\rangle_{i\ge k_1}}$

$\forall\ k_2\ \exists y_2 \in S_{\langle x_i\rangle_{i\ge k_2}}$

$\dots$

$\forall\ k_n\ \exists\ y_n \in S_{\langle x_i\rangle_{i\ge k_n}}$ with $d_b((e_i)_1^n, (y_i)_1^n) < 1+\varepsilon$.

From Krivine's theorem there exists $1 \le p \le \infty$ so that the unit vector basis of ℓ_p^n belongs to $\{X\}_n^t$ for all n. The notation $\{X\}_n^t$ should more precisely be $\{X,(x_i)\}_n^t$ since its structure generally depends on the particular basis one chooses for X. For example James' famous quasi-reflexive space J has two well known bases, (u_i) and (v_i) given isomorphically by:

$$\Big\|\sum a_i u_i\Big\| = \sup\bigg\{\Big(\sum_{i=1}^\infty\Big(\sum_{j\in E_i} a_j\Big)^2\Big)^{1/2} : E_1 < E_2 < \cdots \text{ are intervals in } \mathbb{N}\bigg\}\ ,$$

this basis is spreading and boundedly complete, and

$$\Big\|\sum a_i v_i\Big\| = \sup\bigg\{\Big(\sum (a_{p_i} - a_{q_i})^2\Big)^{1/2} : p_1 < q_1 < p_2 < q_2 < \cdots\bigg\}\ ,$$

this basis is shrinking. One has that if $(e_i)_1^n \in \{J,(v_i)\}_n^t$ then (e_i) is 2-equivalent to the unit vector basis of ℓ_2^n, while $(u_i)_1^n \in \{J,(u_i)\}_n^t$ for all n and $\|\sum_1^n u_i\| = n$.

If (e_i) is a K-asymptotic ℓ_1 basis then for all n all normalized block bases $(x_i)_1^n$ of $(e_i)_n^\infty$ are K-equivalent to the unit vector basis of ℓ_1^n. The condition that every $(x_i)_1^n \in \{X,(e_i)\}_n^t$ is K-equivalent to the unit vector basis of ℓ_1^n is weaker. We need certain separation between the support of x_i and that of x_{i+1}.

[MMT] also constructed a coordinate free version of $\{X\}_n^t$. Let $\mathrm{cof}(X)$ denote the set of all finite co-dimensional subspaces of X. Then $(e_i)_1^n \in \{X\}_n$ if

$\forall\ \varepsilon > 0\ \forall\ X_1 \in \mathrm{cof}(X)\ \exists\ x_1 \in S_{X_1}$

$\forall\ X_2 \in \mathrm{cof}(X)\ \exists\ x_2 \in S_{X_2}$

$\dots$

$\forall\ X_n \in \mathrm{cof}(X)\ \exists\ x_n \in S_{X_n}$ with $d_b((e_i)_1^n, (x_i)_1^n) < 1+\varepsilon$.

It is easy to see that $(e_i)_1^n$ must be a monotone basic sequence, an element of $\mathcal{M}_n \equiv$ set of all monotone basic sequences of length n. $\mathcal{M}_n$ is a compact metric space under $d_b(\cdot,\cdot)$ (actually $\log d_b(\cdot,\cdot)$).

$\{X\}_n$ can be understood in terms of a two player game, much like Gowers' game. (S) chooses $X_1 \in \mathrm{cof}(X)$ and (V) chooses $x_1 \in S_{X_1}$ and plays alternate thusly: $(X_1, x_1, \dots, X_n, x_n)$.

$(e_i)_1^n \in \{X\}_n$ if for all $\varepsilon > 0$, (V) has a winning strategy for choosing $(x_i)_1^n$ with $d_b((x_i)_1^n, (e_i)_1^n) < 1+\varepsilon$.

Or one can describe $\{X\}_n$ as the smallest compact subset of $\mathcal{M}_n$ so that, given $\varepsilon > 0$, (S) has a winning strategy to force (V) to select $(x_i)_1^n$ with

$$d_b((x_i)_1^n, \{X\}_n) < 1 + \varepsilon \ .$$

These interpretations are discussed in [MMT], and easily lead to $\{X\}_n \neq \emptyset$ for all n (alternately, we could use spreading models and Rosenthal's ℓ_1 theorem to deduce this).

$\{X, (x_i)\}_n^t$ can also be described in terms of a game like that above where (S) is restricted to choosing tail subspaces, i.e., $[(x_i)_{i \geq k}]$. Also, if (x_i) is a shrinking basis for X it is not hard to see that $\{X, (x_i)\}_n^t = \{X\}_n$ for all n. Moreover it can be shown that given $\varepsilon_n \downarrow 0$ there exists a blocking (F_i) of (x_i) into an FDD (finite dimensional decomposition)

$$F_i = \langle x_j \rangle_{j \in (n_{i-1}, n_i]} \ \text{ for some } \ 0 = n_0 < n_1 < \cdots$$

so that if $n \in \mathbb{N}$ and $(y_i)_1^n$ is a normalized skipped block basis of $(F_i)_n^\infty$ then $d_b((y_i)_1^n, \{X\}_n) < 1 + \varepsilon_n$ ([MMT], [KOS]). By saying $(y_i)_1^n$ is skipped we mean that

$$y_i \in \langle F_j \rangle_{j \in (k_{i-1}, k_i)} \ \text{ for } \ i \leq n \ \text{ for some } \ k_0 < k_1 < \cdots \ ,$$

i.e., "F_{k_i}" is skipped between y_i and y_{i+1}.

When X^* is separable, $\{X\}_n$ can also be described in terms of weakly null trees. We let $T_n = [\mathbb{N}]^{\leq n} = \{(k_i)_1^m : m \leq n,\, k_1 < \cdots k_m \text{ are in } \mathbb{N}\}$ and order T_n by extension. Thus T_n is a countably branching tree of n-levels. A *branch* in T_n is a maximal linearly ordered subset, $\{(k_1), (k_1, k_2), \ldots, (k_1, \ldots, k_n)\}$. A *weakly null n-tree* in X is $(x_\alpha)_{\alpha \in T_n} \subseteq S_X$, ordered by T_n, where each *node* is a weakly null sequence. By *node* we mean the successors to any element of $\phi \cup T_{n-1}$. Thus, for example, $x_{(n)} \xrightarrow{w} 0$ and $w - \lim_{n>2} x_{(2,n)} = 0$.

$\{X\}_n$ is the smallest compact subset of $\mathcal{M}_n$ so that for all $\varepsilon > 0$ and all weakly null n-trees in X there exists a branch, call it $(z_i)_1^n$, with $d_b((z_i)_1^n, \{X\}_n) < 1 + \varepsilon$. Also we note that for every weakly null n-tree $(x_\alpha)_{\alpha \in T_n}$ in X and $\varepsilon > 0$ there exists a full subtree $(y_\alpha)_{\alpha \in T_n}$ and $(e_i)_1^n \in \{X\}_n$ so that every branch $(z_i)_1^n$ satisfies $d_b((z_i)_1^n, (e_i)_1^n) < 1 + \varepsilon$. By *full subtree* we mean that for some $T \subseteq T_n$, $(y_\alpha)_{\alpha \in T_n} = (x_\alpha)_{\alpha \in T}$ and T is order isomorphic to T_n.

We can also describe $\{X, (x_i)\}_n^t$ in terms of trees, $(x_\alpha)_{\alpha \in T_n}$ which are block basis trees w.r.t. (x_i). This just means that each node is a block basis of (e_i).

$\mathcal{A}(X)$ denotes the class of all normalized monotone bases $(e_i)_1^\infty$ so that for all n, $(e_i)_1^n \in \{X\}_n$. Since any element of $\{X\}_n$ extends to one in $\{X\}_{n+1}$, each $(e_i)_1^n \in \{X\}_n$ is the initial part of an *asymptotic version* (an element of $\mathcal{A}(X)$).

$(\{X\}_n)_n$ is closed under taking block bases: if $(e_i)_1^n \in \{X\}_n$ and $(y_i)_1^m$ is a normalized block basis of $(e_i)_1^n$ then $(y_i)_1^m \in \{X\}_m$.

If $(x_i) \subseteq S_X$ is weakly null with spreading model $(\tilde{x}_i)$ then $(\tilde{x}_i) \in \mathcal{A}(X)$. The difference between spreading models and $\{X\}_n$ is that in one case you are given a sequence and pass to a subsequence (x_i) to stabilize $\| \sum_1^n a_i \tilde{x}_i \|$. In the other, for X^* separable, you stabilize the branches of a tree which allows for more outputs. In general the theory for $\{X\}_n$ is more complete than that for spreading models.

Proposition 5.15. [MMT]

1) *For all X either $c_0 \in \mathcal{A}(X)$ or there exists $1 \leq p < \infty$ with $\ell_p \in \mathcal{A}(X)$ (isometrically).*
2) *If $Y \in \mathcal{A}(X)$ and $Z \in \mathcal{A}(Y)$ then $Z \in \mathcal{A}(X)$.*
3) *For all X there exists $Y \subseteq X$ so that for all $Z \subseteq Y$, $\mathcal{A}(Y) = \mathcal{A}(Z)$ (stabilization).*

4) *If* $|\mathcal{A}(X)| = 1$, *or more generally if* $|\{X\}_2| = 1$ *then* X *contains some* ℓ_p, $1 \le p < \infty$ *or* c_0.

Perhaps only 4) requires some discussion. First we have $\{X\}_2 = \ell_p^2$ for some $1 \le p \le \infty$. (This in fact yields $\mathcal{A}(X) = \{\ell_p\}$.) We can inductively, given $\varepsilon_n \downarrow 0$, choose a basic $(x_i) \subseteq S_X$ so that for all n and $y \in S_{\langle x_i \rangle_{n+1}^\infty}$, $d_b((x_n, y), \ell_p^2) < 1 + \varepsilon_n$. Thus

$$\begin{aligned} \Big\| \sum a_i x_i \Big\| &\overset{1+\varepsilon_1}{\sim} \Big(|a_1|^p + \Big\| \sum_{i \ge 2} a_i x_i \Big\|^p \Big)^{1/p} \\ &\overset{(1+\varepsilon_1)(1+\varepsilon_2)}{\sim} \Big(|a_1|^p + |a_2|^p + \Big\| \sum_{i \ge 3} a_i x_i \Big\|^p \Big)^{1/p} \\ &\dots \overset{1+\varepsilon}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} . \end{aligned}$$

Despite the stabilization result in 3) there is an interesting open question concerning $K(X) = \{p : \ell_p \in \mathcal{A}(X)\}$, where if $p = \infty$ we mean $c_0 \in \mathcal{A}(X)$. $K(X)$ is a closed subset of $[1, \infty]$ but can be disconnected; e.g., $K(\ell_p \oplus \ell_2) = \{p, 2\}$. However this is not clear when stabilized.

Problem. Let $\mathcal{A}(X) = \mathcal{A}(Y)$ for all $Y \subseteq X$. Is $K(X)$ an interval?

An example was given in [OS5] of a reflexive space with an unconditional basis so that $K(X) = [1, \infty]$ for a stabilized X. The example of [OS1], mentioned earlier, gives a reflexive space X, stabilized, so that $\mathcal{A}(X)$ is all monotone bases, M_∞.

X is K-*asymptotic* ℓ_p if for all $(y_i) \in \mathcal{A}(X)$, (y_i) is K-equivalent to the unit vector basis of ℓ_p. One might consider the property $[(y_i)] \overset{K}{\sim} \ell_p$ instead but this was shown to be an equivalent definition for $1 < p < \infty$ and remains open for $p = 1$ [MMT].

Similarly, X is K-*asymptotically unconditional* if for all $(y_i) \in \mathcal{A}(X)$, (y_i) is K-unconditional. The reflexive H.I. space with an asymptotic ℓ_1-basis of [AD] shows that we cannot always get an unconditional basis in such a setting. We could do this if it were 1-asymptotically unconditional, as in the proof of 4) above.

If $|\mathcal{A}(X)| = 1$ isomorphically, i.e., $(y_i) \sim (z_i)$ for all $(y_i), (z_i) \in \mathcal{A}(X)$ then for some $1 \le p \le \infty$ and $K < \infty$, $(y_i) \overset{K}{\sim} \ell_p$ for all $(y_i) \in \mathcal{A}(X)$. If no such K existed we could paste together segments from different elements of $\mathcal{A}(X)$ to get $(y_i) \not\sim \ell_p$. Thus X is asymptotic ℓ_p. In this case it is not hard to show that X contains an asymptotic ℓ_p basic sequence. As we mentioned, even if all spreading models of X are 1-equivalent to the unit vector basis of ℓ_p $(1 < p < \infty)$ it is not known if $X \supseteq \ell_p$.

Problem. Assume for all $(\tilde{x}_i) \in SP(X)$, $(\tilde{x}_i) \overset{K}{\sim}$ the unit vector basis of ℓ_p. Does X contain an asymptotic ℓ_p subspace? This is open even for $K = 1$.

Example. [OS6] Let $1 < q < p < \infty$. There exists a reflexive space X with an unconditional basis having the properties

a) For all $(\tilde{x}_i) \in SP(X)$, $(\tilde{x}_i)$ is 1-equivalent to the unit vector basis of ℓ_p.

(5.2) $\left\{ \begin{array}{l} \text{In fact for } \varepsilon > 0 \text{ every normalized weakly null sequence in } X \text{ has} \\ \text{a subsequence } 1+\varepsilon\text{-equivalent to the unit vector basis of } \ell_p. \end{array} \right.$

b) $\ell_q \in \mathcal{A}(X)$.

$X = (\sum X_n)_{\ell_p}$ where we shall define X_n as the completion of $c_{00}(T_n)$ under

$$\|x\|_n = \sup\left\{\left(\sum_1^m \|x|_{\beta_i}\|_q^p\right)^{1/p} : (\beta_i)_1^m \text{ are disjoint segments in } T_n\right\}.$$

A *segment* in T_n is an interval $[\alpha, \beta] = \{\gamma : \alpha \leq \beta \leq \gamma\}$. The coordinate functionals $(e_\alpha)_{\alpha \in T_n}$, given by $e_\alpha(\gamma) = \delta_{\alpha,\gamma}$, form a 1-unconditional basis for X_n. To prove a), it suffices to show that each X_n satisfies (5.2) and then it is easy to argue that X satisfies (5.2). Indeed if Y_n has (5.2) for all n so does $(\sum Y_n)_{\ell_p}$. Now this fact plus an easy induction argument yields each X_n has (5.2).

To see b) we note that if $(e_{\alpha_i})_1^n$ is any branch of X_n then $(e_{\alpha_i})_1^n$ is 1-equivalent to the unit vector basis of ℓ_q^n. $\square$

Some results do exist for finding asymptotic ℓ_p subspaces of a space X.

Proposition 5.16. [JKO] *(see also* [Sa2]*) Let $1 \leq p < \infty$ and let X have a basis (x_i). Assume that for some $K < \infty$ for all n and $(e_i)_1^n \in \{X\}_n^t$, $\|\sum_1^n e_i\| \overset{K}{\sim} n^{1/p}$. Then X contains an asymptotic ℓ_p subspace.*

The result is trivial for $p = \infty$ since $(\pm e_i)_1^n \in \{X\}_n^t$ whenever $(e_i)_1^n \in \{X\}_n^t$.

Proposition 5.16 was used in proving

Theorem 5.17. [Tc] *Let X satisfy: For all $Y \subseteq X$ there exists $M_Y < \infty$ so that for all n there exist $U_1, U_2, \ldots, U_n \subseteq Y$ with $\|\sum_1^n y_i\| \overset{M_Y}{\sim} \|\sum_1^n x_i\|$ for all $y_i, x_i \in U_i$, $\|x_i\| = \|y_i\|$ for $i \leq n$. Then X contains an asymptotic ℓ_p subspace for some $1 \leq p \leq \infty$.*

In fact X contains a *strongly asymptotic ℓ_p basic sequence* (e_i). This means that for some K and all n if $(x_i)_1^n \subseteq S_{\langle e_i \rangle_{i \geq n}}$ with $\text{supp}(x_i) \cap \text{supp}(x_j) = \emptyset$ for $i \neq j$ then $\|\sum_1^n a_i x_i\| \overset{K}{\sim} (\sum_1^n |a_i|^p)^{1/p}$.

The unit vector basis of T (T^*) is strongly asymptotic ℓ_1 (c_0) [CO]. Strongly asymptotic ℓ_p bases were studied with regard to minimality in [DFKO] where it was shown that if X has a normalized strongly asymptotic ℓ_p basis (e_i) and X is minimal then

- For $1 \leq p < 2$, $X \hookrightarrow \ell_p$.
- For $2 \leq p < \infty$, (e_i) is equivalent to the unit vector basis of ℓ_p.

Problem. Does there exist for $1 \leq p < \infty$ a minimal space X with an asymptotic ℓ_p basis which does not embed into ℓ_p?

Recall T^* is minimal and has an asymptotic c_0-basis but of course does not embed into c_0.

While we are discussing minimality again we should mention a very nice result of A. Pelczar [Pel].

Theorem 5.18. *Suppose that X is saturated with subsymmetric bases, i.e., for all $Y \subseteq X$, Y contains a subsymmetric basic sequence. Then Y contains a minimal subspace.*

Other asymptotic notions have been studied. For example, stability, which we mentioned earlier, is an asymptotic notion. An isomorphic version of stability is the following condition.

Definition. X is C-*asymptotically symmetric* (a.s.) if for all $m \in \mathbb{N}$, for all bounded sequences $(x_n^i)_{n=1}^\infty \subseteq X$, $i \le m$, and for all permutations σ of $\{1, 2, \dots, m\}$

$$\lim_{n_1 \to \infty} \dots \lim_{n_m \to \infty} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\| \le C \lim_{n_{\sigma(1)} \to \infty} \dots \lim_{n_{\sigma(m)} \to \infty} \Big\| \sum_{i=1}^m x_{n_i}^i \Big\|$$

provided these iterated limits exist.

If X is stable it is 1-asymptotically symmetric. If X is a.s. then all spreading models are uniformly symmetric, i.e., K-symmetric for some fixed $K < \infty$.

This condition and variations of it were studied in [JKO]. For example it was shown that if every normalized weakly null sequence in X has a subsequence equivalent to the unit vector basis of c_0 then X is w.a.s. (weakly a.s.). X is w.a.s. if the defining condition for a.s. is restricted to weakly convergent sequences.

We will discuss asymptotic structure some more in the next section. Here are a few spaces where it is easy to characterize $\mathcal{A}(X)$ and $SP(X)$.

1) $1 < p < \infty$; $\mathcal{A}(\ell_p) = SP(\ell_p) = \{(e_i)\}$ where (e_i) is the unit vector basis of ℓ_p.
2) $\mathcal{A}(\ell_1)$ is the same as in 1), just the unit vector basis of ℓ_1. $SP(\ell_1)$ consists of ℓ_1-bases, but not isometrically. For example $(\frac{1}{2}(e_1 + e_n))_{n>1} \in SP(\ell_1)$.
3) $\mathcal{A}(c_0)$ is just the c_0 basis, again but $SP(X)$ contains the c_0-basis (isometrically) as well as bases equivalent to (s_i).
4) $\mathcal{A}(C[0,1]) = \mathcal{M}_\infty$. $SP(C[0,1]) =$ all 1-spreading bases.
5) For $2 < p < \infty$, $\mathcal{A}(L_p)$ consists, up to a constant K_p, of all block bases of the Haar basis. $SP(L_p)$ consists of the ℓ_p basis (isometrically) and bases equivalent to the unit vector basis of ℓ_2.
6) $\mathcal{A}(J)$ and $SP_w(J)$ consists of normalized bases 2-equivalent to the unit vector bases of ℓ_2. $SP(J)$ contains more, such as the boundedly complete basis (u_i) for J discussed earlier.
7) $SP_w(C(\omega^\omega))$ is all spreading models of any normalized weakly null basic sequence [O3].

It is not hard to see that this set coincides with the set of all normalized suppression -1 unconditional bases, call this $SU(1)$.

Indeed if $(u_i) \in SU(1)$ define Y to be the completion of c_{00} under

$$\|(a_i)\| = \sup \left\{ \Big\| \sum_{i \in F} a_i u_i \Big\| : F \in S_1 \right\} .$$

Y is normed by $\mathcal{F} = \{y^*|_F : F \in S_1,\ y^* \in B_{U^*}\}$ so $Y = \overline{(c_{00}, \|\cdot\|_{\mathcal{F}})}$. The unit vector basis (e_i) for Y has $(\tilde{e}_i) = (u_i)$. $Y \overset{1}{\hookrightarrow} C(\mathcal{F})$ where $\mathcal{F}$ is compact metric (ω^*) and by Cantor-Bendixson index arguments, $Y \overset{1}{\hookrightarrow} C(\omega^\omega)$.

Problem. If $SP_w(X) \supseteq SU(1)$ does $c_0 \hookrightarrow X$? Does $C(\omega^\omega) \hookrightarrow X$?

6. Embedding Theorems

We have seen that every X embeds isometrically into $C[0,1]$, a space with a monotone basis. Thus every X can be put into a space with a coordinate system. Our main topic in this section is results of a type # 3 problems (section 1). To take an X with some property (P) and embed it into a coordinatized space reflecting the property. We will also be interested in finding universal spaces Y for the property (P), i.e., every X with (P) embeds into Y and Y has (P) or perhaps a weaker version of (P).

Pełczyński [Pe3] proved a universal result for bases and unconditional bases. Schechtman [Sch] gave a simpler proof. We will refer to his argument again later.

Theorem 6.1.

a) *There exists a space V with a normalized unconditional basis (e_i) so that every normalized unconditional basis is equivalent to some subsequence of (e_i).*
b) *There exists a space B with a normalized basis (e_i) so that every normalized basis is equivalent to a subsequence of (e_i), which moreover is naturally complemented in B.*

Proof. a) Let (x_i) be dense in $S_{C[0,1]}$. Define $U = \overline{(c_{00}, \|\cdot\|)}$ where

$$\|(a_i)\| = \sup\left\{\left\|\sum \varepsilon_i a_i x_i\right\| : \varepsilon_i \pm 1\right\}$$

(e_i) is a 1-unconditional basis for U and if (u_i) is any normalized unconditional basic sequence then $[(u_i)] \overset{1}{\hookrightarrow} C[0,1]$. So we can choose (x_{n_i}) with $\sum \|u_i - x_{n_i}\| < \varepsilon$ and obtain the result, $(u_i) \sim (e_{n_i})$.

b) We choose (x_i) as in a) and construct a tree $(x_\alpha)_{\alpha\in T_\infty} \subseteq S_{C[0,1]}$ from this as follows. Here $T_\infty = \bigcup_{n\in\mathbb{N}} T_n$ is the countably branching tree of countably many levels. We let $x_{(n)} = x_n$ and if $(x_{(\alpha)}$ is defined where $\alpha = (n_1, \dots, n_k)$ then $x_{(\alpha,m)} = x_m$ for $m > n_k$. B is the completion of $(c_{00}(T_\infty), \|\cdot\|)$ where $\|(a_\alpha)\| = \sup\{\|\sum_{\alpha\in I} a_\alpha x_\alpha\| : I$ is a segment in T_∞. If (y_i) is any normalized basic sequence we can choose a branch $(\alpha_i)_1^\infty$ of T_∞ with $\sum \|y_i - x_{\alpha_i}\| < \varepsilon$ and obtain $(y_i) \sim (e_{\alpha_i})$. Also

$$P\Big(\sum a_\alpha e_\alpha\Big) = \sum_{i=1}^\infty a_{\alpha_i} e_{\alpha_i} \quad \text{is a norm 1 projection onto} \quad [(e_{\alpha_i})]\ .$$

□

There is an embedding result due to Lindenstrauss [Li] which can be proved by an argument we can view as a precursor of the Maurey-Rosenthal example.

Theorem 6.2. *Every X with an unconditional basis is complemented in a space with a symmetric basis.*

Sketch of proof. Let (u_i) be a normalized 1-unconditional basis for X. We will define a norm $\|\cdot\|$ on c_{00} so that if $Y = \overline{(c_{00}, \|\cdot\|)}$ then (e_i) is a 1-symmetric basis for Y. Moreover there exists a block basis (x_i) of (e_i) of the form

$$x_i = \frac{\mathbf{1}_{E_i}}{\sqrt{|E_i|}} \quad \text{where} \quad E_1 < E_2 < \cdots \quad \text{so that} \quad (x_i) \sim (u_i)\ .$$

The complementation will follow from the fact that "averaging projections" in a space with symmetric basis are always bounded:

$$P(a_n) = \sum_{i=1}^{\infty} \left(\frac{\sum\limits_{n \in E_i} a_n}{|E_i|} \right) \sum_{n \in E_i} e_n \ .$$

To define $\|\cdot\|$ we let $n_i \uparrow \infty$ rapidly and choose $E_1 < E_2 < \cdots$ in $\mathbb{N}$ with $|E_i| = n_i$. We let

$$\left\{ \mathcal{F} = \sum_{1}^{\infty} b_i \mathbf{1}_{E_i} / \sqrt{|E_i|} : \| \sum b_i u_i^* \| \le 1 \right\}$$

and set

$$\|x\| = \|x\|_\infty \vee \sup \{ |\langle f, x_\pi \rangle| : f \in \mathcal{F} \ \text{ and } \ \pi \ \text{ is any permutation of } \ \mathbb{N} \} \ .$$

The argument proceeds much like the Maurey-Rosenthal example. □

This result has been extended to where X and the superspace are both reflexive [Sz] or even uniformly convex [Da].

We will need the following non-trivial theorem later.

Theorem 6.3 ([Z],[DFJP])**.**

a) *If X is reflexive, X embeds into a reflexive space with a basis.*
b) *If X^* is separable, X embeds into a space with a shrinking basis.*
c) *If X^* is separable, X is a quotient of a space with a shrinking basis.*

We will also need to expand our notion of a coordinate system from a basis to a finite dimensional decomposition (FDD).

Definition. A sequence of finite dimensional subspaces (F_n) of X is an FDD for X if each $F_n \ne \{0\}$ and for all $x \in X$ there exist unique vectors $x_n \in F_n$ with $x = \sum_1^\infty x_n$.

The results of bases carry over. Non-zero (F_n) form an FDD for X if $[\langle F_n : n \in \mathbb{N} \rangle] = X$ and $\sup_{n \le m} \|P_{[n,m]}\| < \infty$ where for $x = \sum x_i$, $x_i \in F_i$, $P_{[n,m]} x = \sum_{i=n}^m x_i$. $\sup_{n \le m} \|P_{[n,m]}\|$ is called the *projection constant* of the FDD (F_n) and (F_n) is *bimonotone* if this equals 1. One can also talk about unconditional, shrinking and boundedly complete FDD's and make the same observations we made regarding bases. Not every X has an FDD and a space can have an FDD but not a basis [Sza].

(G_i) is a *blocking* of the FDD (F_i) for X if for some $n_0 = 0 < n_1 < \cdots$, $G_i = \langle F_j \rangle_{j \in (n_{i-1}, n_i]}$ and in this case (G_i) is an FDD for X with projection constant not exceeding that of (F_i).

If (F_n) is a sequence of finite dimensional Banach spaces then it forms a 1-unconditional FDD for $(\sum F_n)_{\ell_p}$, a reflexive space when $1 < p < \infty$. The norm of $\sum x_n$, $x_n \in F_n$, is given by $(\sum \|x_n\|^p)^{1/p}$. Our next embedding result will be to characterize when a reflexive X embeds into such a space.

What property should such an X have? Well certainly every normalized weakly null sequence in X must have a subsequence $K + \varepsilon$-equivalent to the unit vector basis of ℓ_p if $X \overset{K}{\hookrightarrow} (\sum F_n)_{\ell_p}$. But this is not enough as the example in Section 5 reveals (there exists a reflexive X with this property but $\ell_q \in \mathcal{A}(X)$ for some $1 < q < p$, and $\mathcal{A}((\sum F_n)_{\ell_p}) = \{\text{the unit vector basis of } \ell_p\}$). However X would have a stronger property: Every weakly null tree $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ admits a branch $K + \varepsilon$-equivalent to the unit vector basis of ℓ_p. Note that if (x_n) is a sequence and we set $x_{(n_1,\ldots,n_k)} = x_{n_k}$ then the branches of $(x_\alpha)_{\alpha \in T_\alpha}$

are the subsequences of (x_n) and thus the tree condition is a stronger hypothesis. It turns out that this is the right condition [OS6].

Theorem 6.4. *Let $1 < p < \infty$ and let X be reflexive and assume that every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p. Then X embeds into $(\sum F_n)_{\ell_p}$ for some sequence of finite dimensional spaces (F_n).*

In the last section we discussed $\{X\}_n$ in terms of weakly null trees of length n (for X^* separable) as well as in terms of the 2 player $(S),(V)$ game. That theory was quite tight in that $\{X\}_n$ was a certain compact subset of $\mathcal{M}_n$. If we play the $(S),(V)$ game forever we have the *infinite asymptotic game* with plays

$$(X_1, x_1, X_2, x_2, \ldots),$$
$$X_i \in \mathrm{cof}(X) \text{ and } x_i \in S_{X_i}.$$

Unlike the case of $\{X\}_n$ there is generally no $\{X\}_\infty$ analogue; i.e., no smallest set $\{X\}_\infty \subseteq \mathcal{M}_\infty$ so that (V) can always choose (up to ε) any $(x_i) \in \{X\}_\infty$ or (S) can always force (V) to select (x_i) almost in $\{X\}_\infty$. We don't have the compactness of $\mathcal{M}_n$ to make the argument work. But we can begin with a class $\mathcal{A}$ of normalized basic sequences, e.g., those equivalent to the ℓ_p basis, and postulate that every weakly null tree in X has a branch in $\mathcal{A}$ and see where this leads.

We discuss this in a more general setting: the $(\mathcal{A},\beta)$ game in $Z = [(E_i)]$. In this first setting we assume our space has an FDD and consider the analogue of the infinite asymptotic game with respect to the FDD, much like $\{X\}_n^t$.

Let Z be a space with an FDD (E_i) and let $\mathcal{A} \subseteq S_Z^\omega$, a set of normalized sequences in Z. Let $B_i \subseteq S_Z$ for $i \in \mathbb{N}$ and $\beta = \prod_1^\infty B_i$. (S) and (V) make these plays: (S) selects $n_1 \in \mathbb{N}$ and (V) selects $x_1 \in S_{\langle E_i\rangle_{i\ge n_1}} \cap B_1$. At the k^{th}-play (S) selects n_k and (V) chooses $x_k \in S_{\langle E_i\rangle_{i\ge n_k}} \cap B_k$ and plays continue forever. We will declare (V) the winner if $(x_i)_1^\infty \notin \mathcal{A}$ and otherwise (S) wins. Now this is analysis so we need to allow for perturbations to get a useful theory.

If $\bar\varepsilon = (\varepsilon_i) \subseteq (0,1)$ we let

$$\mathcal{A}_{\bar\varepsilon} = \{(z_i) \in S_Z^\omega : \|z_i - x_i\| \le \varepsilon_i \text{ for all } i \text{ and some } (x_i) \in \mathcal{A}\} .$$

$\bar{\mathcal{A}}$ is the closure of $\mathcal{A}$ in S_Z^ω under the product topology of the discrete topology on S_Z. Thus $(z_i) \in \bar{\mathcal{A}}$ if for all n there exists $(x_i^n) \in \mathcal{A}$ with $x_i^n = z_i$ for $i \le n$. So if $\mathcal{A}$ consisted, for example, of all normalized block sequences of (E_i) that were K-equivalent to the unit vector basis of ℓ_p then given $\varepsilon > 0$ we could choose $\varepsilon_i \downarrow 0$ so that $\bar{\mathcal{A}}_{\bar\varepsilon}$ contained only sequences $K+\varepsilon$ equivalent to the unit vector basis of ℓ_p

In this $(\mathcal{A},\beta)$ game we will write $W_{(V)}(\mathcal{A},\beta)$ if (V) has a winning strategy and similarly $W_{(S)}(\mathcal{A},\beta)$ means (S) has a winning strategy. We should note that if $\mathcal{A}$ is Borel in the product topology of the discrete topology, as it will be in our applications, the game is determined [Ma], i.e., either (V) or (S) has a winning strategy.

One can think of what it means for the game to be determined in this manner.

(V) has a winning strategy means that:
$\forall\, n_1\ \exists\, x_1 \in S_{\langle E_i\rangle_{i\ge n_1}} \cap \beta_1$
$\forall\, n_2\ \exists\, x_2 \in S_{\langle E_i\rangle_{i\ge n_2}} \cap \beta_2 \ldots$
so that $(x_i) \notin \mathcal{A}$.

(S) has a winning strategy means that:
$\exists\, n_1\ \forall\, x_1 \in S_{\langle E_i\rangle_{i\ge n_1}} \cap \beta_1$

$\exists\, n_2\ \forall\, x_2 \in S_{\langle E_i\rangle_{i\ge n_2}} \cap \beta_2 \ldots$
$(x_i) \in \mathcal{A}$

Formally these are the negations of each other but they are infinite sentences so, unless the game is determined, we cannot say one or the other is necessarily true.

We recall one more definition before stating our next combinatorial result.

(G_i) is a *skipped blocking* of (E_i) if there exist integers $n_1 \le m_1 < m_1+1 < n_2 \le m_2 < m_2+1 < n_3 \le m_3 < \cdots$ so that $G_i = \langle E_j\rangle_{j\in[n_i,m_i]}$. The key element here is that at least one E_j is skipped between G_i and G_{i+1}. (z_i) is a *skipped block sequence* of (E_i) if $z_i \in G_i$ for some skipped blocking (G_i) of (E_i).

Proposition 6.5. *Assume that for all $\bar\varepsilon$, $W_{(S)}(\bar{\mathcal{A}}_{\bar\varepsilon}, \beta)$. Then for all $\bar\varepsilon$ there exists a blocking (G_i) of (E_i) so that every skipped block sequence (z_i) of (G_i), with $z_i \in B_i$ for all i, is in $\bar{\mathcal{A}}_{\bar\varepsilon}$.*

Note how the conclusion provides an easy winning strategy for (S): at the $(i+1)^{st}$ play, just choose n_{i+1} skipping some G_j past the support of z_i.

Now if X in Theorem 6.4 had an FDD we could use this, and some tricks to come, to deduce the theorem. However we only have $X \subseteq Z$, a reflexive space with an FDD (by Zippin's Theorem 6.3). So we must broaden our game and the combinatorics.

Let $X \subseteq Z$, a space with an FDD (E_i) and let $\mathcal{A} \subseteq S_X^\omega$. We define the $(\mathcal{A}, Z)$ *game* as follows. Set $X_n = [(E_i)_{i\ge n}] \cap X$, a finite codimensional subspace of X.

(S) chooses $n_1 \in \mathbb{N}$ and (V) chooses $x_1 \in S_{X_{n_1}}$. Plays alternate forever thusly as before. (S) wins if $(x_i) \in \mathcal{A}$.

So $W_{(V)}(\mathcal{A}, Z)$ just means that there exists a tree $(x_\alpha)_{\alpha\in T_\infty} \subseteq S_X$ so that for all $\alpha = (n_1,\ldots,n_\ell) \in T_\infty \cup \{\emptyset\}$ $(x_{\alpha,n}) \in X_n$ if $n > n_\ell$ and no branch of the tree lies in $\mathcal{A}$.

If the game is determined then $W_{(S)}(\mathcal{A}, Z)$ means the negation of the above, namely, for all trees $(x_\alpha)_{\alpha\in T_\infty} \subseteq S_X$, as above, some branch lies in $\mathcal{A}$.

Now assuming $W_{(S)}(\mathcal{A}, Z)$ we should like to find a blocking (G_j) of (E_i) so that skipped block sequences of G_j lie almost in $\mathcal{A}$ but a problem is that these need not lie in X so we need a definition.

Definition. Let $\bar\delta = (\delta_n) \subseteq (0,1)$. $(z_j) \subseteq S_Z$ is a $\bar\delta$*-skipped block sequence* of (E_n) if there exist integers $1 \le k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ so that $\|z_n - P^E_{(k_n,\ell_n]} z_n\| < \delta_n$ for all n.

We also will need a technical condition concerning the embedding of X into Z.

So let $X \subseteq Z$, a space with an FDD (E_i).

$$\text{There exists } C>0 \text{ so that for all } m\in\mathbb{N} \text{ and } \varepsilon>0 \text{ there exists}$$
$$\text{an integer } n=n(\varepsilon,m)>m \text{ with } \|x\|_{X/X_m} \le C\left[\|P^E_{[1,n]}(x)\| + \varepsilon\right] \tag{6.1}$$

It can be shown that (6.1) holds if (E_i) is a shrinking FDD for Z or if (E_i) is boundedly complete, in which case Z is a dual space naturally, and if B_X is weak*-closed in Z.

This just means that if $x^n = \sum x_i^n \in B_X$, $x_i^n \in E_i$, and $x_i^n \to x_i$ as $n\to\infty$ for all i, then $\sum x_i \in X$.

If $\mathcal{A} \subseteq S_X^\omega$ we define $\overline{\mathcal{A}^X_{\bar\varepsilon}}$ as before but restricted to sequences in X.

Theorem 6.6. *Let Z have an* FDD(E_i) *with projection constant K. Assume $X \subseteq Z$ and* (6.1) *is satisfied. Let $\mathcal{A} \subseteq S_X^\omega$. Then the following are equivalent.*

a) *For all $\bar\varepsilon = (\varepsilon_n)$, $\varepsilon_n \downarrow 0$, $W_{(S)}(\overline{\mathcal{A}^X_{\bar\varepsilon}}, Z)$*

b) *For all* $\bar\varepsilon = (\varepsilon_n)$, $\varepsilon_n \downarrow 0$, *there exists a blocking* (G_n) *of* (E_i) *so that every* $\bar\varepsilon/420K$*-skipped block sequence* $(z_n) \subseteq S_X$ *of* (G_n) *satisfies* $(z_n) \in \overline{\mathcal{A}^X_{\bar\varepsilon}}$.

If X^* *is separable we can add*

c) *For all* $\bar\varepsilon = (\varepsilon_n)$, $\varepsilon_n \downarrow 0$, *every weakly null tree in* S_X *admits a branch in* $\overline{\mathcal{A}^X_{\bar\varepsilon}}$.

If (E_i) *is boundedly complete and* B_X *is weak*-closed in* Z *we can add*

d) *For all* $\bar\varepsilon = (\varepsilon_n)$, $\varepsilon_n \downarrow 0$, *every weak*-null tree in* S_X *has a branch in* $\overline{\mathcal{A}^X_{\bar\varepsilon}}$.

Note. Weak*-null trees are defined just like weakly null trees: the nodes are weak*-null.

Remark. We began with the infinite asymptotic game and then switched to discussing the game for $X \subseteq Z = [(E_i)]$ where (S) choose tail subspaces, X_n. If (E_i) is shrinking these games are the same in the sense that (S) has a winning strategy for all $\bar{\mathcal{A}}_{\bar\varepsilon}$ in one game iff it has a winning strategy in the other. More generally using arguments of [JRZ] any X can be placed inside a Z with an FDD (E_i) where the two games are again equivalent in the above sense [OS7]. Moreover if (S) wins, given $\bar\varepsilon$ one can find a blocking (G_i) of (E_i) so that every $\bar\varepsilon/420K$-skipped block sequence in S_X w.r.t. (G_j) is in $\overline{\mathcal{A}^X_{\bar\varepsilon}}$.

The use of these $\bar\delta$-skipped blocking results in proving Theorem 6.4 and other embedding theorems to follow will make essential use of the following result of W.B. Johnson [Jo2].

Lemma 6.7. *Let* Z *have a boundedly complete* FDD(E_i) *with projection constant* K *and assume* $X \subseteq Z$ *is such that* B_X *is weak*-closed in* Z. *Let* $\delta_i \downarrow 0$. *There exists a blocking* (F_i) *of* (E_i) *given by* $F_i = \langle E_j \rangle_{j \in (N_{i-1}, N_i]}$ *for some* $0 = N_0 < N_1 < \cdots$ *with the following properties.*

For all $x \in S_X$ *there exists* $(x_i) \subseteq X$ *and for all* i *there exists* $t_i \in (N_{i-1}, N_i)$ *(*$t_0 = 0$ *and* $t_1 > 1$*) satisfying*

a) $x = \sum_j x_j$

b) $\|x_i\| < \delta_i$ *or* $\|P^E_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i \|x_i\|$

c) $\|P^E_{(t_{i-1},t_i)} x - x_i\| < \delta_i$

d) $\|x_i\| < K+1$

e) $\|P^E_{t_i} x\| < \delta_i$

Moreover the above holds for any further blocking of (F_i) *(which redefines the* N_i*'s).*

Remark. Thus if $x \in B_X$ we can write $x = \sum_1^\infty x_i$, $(x_i) \subseteq X$ where if $B = \{i : \|x_i\| \ge \delta_i\}$ then $(x_i/\|x_i\|)_{i \in B}$ is a $\bar\delta$-skipped block sequence w.r.t. (E_i). Moreover the skipped blocks (E_{t_i}) are in predictable intervals, $t_i \in (N_{i-1}, N_i)$.

Proof. For all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n > N$ such that if $x \in B_X$, $x = \sum y_i$, $y_i \in E_i$ then there exists $t \in (N, n)$ with

$$\|y_t\| < \varepsilon \ \text{ and } \ \operatorname{dist}\Big(\sum_1^{t-1} y_i, X\Big) < \varepsilon \ .$$

To prove this we assume not and obtain $y^{(n)} \in B_X$ for $n > N$ failing the conclusion for $t \in (N, n)$. Choose $y^{(n_i)} \xrightarrow{\omega^*} y \in X$ and let $t > N$ satisfy $\|P^E_{[t,\infty)} y\| < \varepsilon/2K$. Choose $y^{(n)}$ from the subsequence (y^{n_i}) so that $\|P^E_{[1,t]}(y - y^{(n)})\| < \varepsilon/2K$. Then

$$\|P^E_{[1,t)} y^{(n)} - y\| \le \|P^E_{[1,t)}(y^{(n)} - y)\| + \|P^E_{[t,\infty)} y\| < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} < \varepsilon \ .$$

Also

$$\|P_t^E y^{(n)}\| \le \|P_t^E(y^{(n)} - y)\| + \|P_t^E y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

This contradicts our choice of $y^{(n)}$.

Let $\varepsilon_i \downarrow 0$ and by this observation we can choose $0 = N_0 < N_1 < \cdots$ so that for all $x \in S_X$ there exists $t_i \in (N_{i-1}, N_i)$ and $z_i \in X$ with $\|P_{t_i}^E x\| < \varepsilon_i$ and $\|P_{[1,t_i)}^E x - z_i\| < \varepsilon_i$ for all i. Set $x_1 = z_1$, $x_i = z_i - z_{i-1}$ for $i > 1$. Then $\sum_{i=1}^n x_i = z_n \to x$ so a) holds. Also,

$$\|P_{(t_{i-1},t_i)}^E x - x_i\| \le \|P_{[1,t_i)}^E x - z_i\| + \|P_{[1,t_{i-1}]}^E x - z_{i-1}\| < \varepsilon_i + 2\varepsilon_{i-1}$$

and

$$\|P_{(t_{i-1},t_i)}^E x_i - x_i\| = (I - P_{(t_{i-1},t_i)}^E)(x - P_{(t_{i-1},t_i)}^E x)\| < (K+1)(\varepsilon_i + 2\varepsilon_{i-1}) \ .$$

b), c) and d) follow if $(K+1)(\varepsilon_i + 2\varepsilon_{i-1}) < \delta_i^2$. □

Sketch of proof of Theorem 6.4.

Recall we have $X \subseteq Z$, a reflexive space with an FDD. Every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p. We first note that there exists a $K < \infty$ so that every such tree admits a branch K-equivalent to the unit vector basis of ℓ_p. If not then by the game interpretation (V) could successively apply winning strategies to choose $(x_1, \ldots, x_{n_1}, \ldots, x_{n_2}, \ldots)$ so that $(x_{n_{i-1}+1}, \ldots, x_{n_i})$ is not i-equivalent to the unit vector basis of ℓ_p so $(x_i)_1^\infty$ is not equivalent to the unit vector basis of ℓ_p.

So we can find a blocking (G_i) of (E_i) so that every $\bar{\delta}$-skipped block sequence in S_X w.r.t. (G_i) is $2K$-equivalent to the unit vector basis of ℓ_p. Then we apply Lemma 6.7 to obtain (N_i) and the blocking (F_i) of (G_i) determined by (N_i).

The claim is then that, naturally, $X \hookrightarrow (\sum F_n)_{\ell_p}$. We have that up to a constant, say $3K$, if $x \in B_X$ and $x = \sum x_i$ as in Lemma 6.7

$$\|x\| \overset{3K}{\sim} \Big(\sum \|x_i\|^p\Big)^{1/p} \ .$$

But if $x = \sum y_i$, $y_i \in F_i$ then

$$x_i \approx P_{(t_{i-1},t_i)}^E (y_{i-1} + y_i)$$

and

$$y_i \approx P_{(N_{i-1},N_i]}^E (x_i + x_{i+1})$$

from which $\|x\| \overset{C(K)}{\sim} (\sum \|y_i\|^p)^{1/p}$. □

Our next result generalizes this to upper ℓ_q, lower ℓ_p estimates. First we need some definitions.

Definition. Let $1 \le q \le p \le \infty$, $C < \infty$. An FDD (F_n) satisfies C-(p,q)-*estimates* if for all block sequences (x_n) of (F_n)

$$C^{-1}\Big(\sum \|x_n\|^p\Big)^{1/p} \le \Big\|\sum x_n\Big\| \le C\Big(\sum \|x_n\|^q\Big)^{1/q} \ .$$

Definition. X satisfies C-(p,q) *tree estimates* if for all weakly null trees in S_X there exist branches (y_i) and (z_i) satisfying:

$$C^{-1}\Big(\sum |a_i|^p\Big)^{1/p} \le \Big\|\sum a_i y_i\Big\| \quad \text{and} \quad \Big\|\sum a_i z_i\Big\| \le C\Big(\sum |a_i|^q\Big)^{1/q}$$

for all $(a_i) \subseteq \mathbb{R}$.

Theorem 6.8. [OS8] *Let X be reflexive, $1 \le q \le p \le \infty$. The following are equivalent*

a) *X satisfies (p,q)-tree estimates*
b) *X is isomorphic to a subspace of a reflexive space Z with an* FDD *satisfying (p,q)-estimates*
c) *X is isomorphic to a quotient of a reflexive space Z having an* FDD *satisfying (p,q)-estimates.*

This presents a difficulty not present in Theorem 6.4 (where $p=q$). We cannot just take $(\sum F_n)_{\ell_p}$ as we did before. How do we get a norm on some blocking (F_i) of an FDD for Z which gives both the lower ℓ_p and upper ℓ_q estimates? Well the lower ℓ_p estimate is easier.

Definition. Let $F=(F_n)$ be an FDD. $Z_p(F)$ is the completion of $\langle (F_n)\rangle$ under $\|\cdot\|_{Z_p}$ given as follows. Let $y=\sum y_n$, $y_n \in F_n$ for all n, $y\in \langle (F_n)\rangle$

$$\|y\|_{Z_p} = \sup\left\{ \left(\sum_{j=1}^\infty \Big\| \sum_{i=n_{j-1}+1}^{n_j} y_i \Big\|^p \right)^{1/p} : 0=n_0<n_1<\cdots \right\}.$$

(F_n) is a bimonotone FDD for $Z_p(F)$ and it is easy to see it satisfies 1-$(p,1)$-estimates. By the arguments like those used in the proof of Theorem 6.4 we can prove a) in

Theorem 6.9. *Let X be reflexive, $1<p<\infty$ and assume X satisfies $(p,1)$-tree estimates.*

a) *Let $X\subseteq Z$, a reflexive space with an* $\mathrm{FDD}(E_i)$. *There exists a blocking $F=(F_i)$ of (E_i) so that X naturally embeds into the reflexive space $Z_p(F)$.*
b) *X is the quotient of a reflexive space with an* FDD *satisfying $(p,1)$-estimates.*

b) plays a key role in the proof of Theorem 6.8. Here are the steps in that proof. X is reflexive.

1) Let $1\le q<\infty$, $1/q+1/q'=1$. If X satisfies (∞,q)-tree estimates then X^* satisfies $(q',1)$-tree estimates.
2) If $F=(F_i)$ is a bimonotone FDD for Z, $1<q\le p<\infty$, $C<\infty$ and (F_i) satisfies C-(∞,q)-estimates in Z then (F_i) satisfies C-(p,q)-estimates in $Z_p(F)$.
3) [Pr] Let (E_i) be an FDD for a reflexive space Z, $1\le q\le p\le\infty$. (E_i) satisfies (p,q)-estimates iff the FDD (E_i^*) for Z^* satisfies (q',p')-estimates where $1/q+1/q'=1$, $1/p+1/p'=1$.

So let X satisfy (p,q)-tree estimates, X reflexive. X^* satisfies $(q',1)$-tree estimates by 1). Thus by Theorem 6.9 b) X^* is a quotient of a reflexive space Z^* with an FDD $F^*=(F_i^*)$ satisfying $(q',1)$-estimates. So X embeds into Z, a reflexive space with an FDD (F_i) satisfying (∞,q)-estimates. By Theorem 6.9 a), X embeds into $Z_p(G)$ for some blocking G of (F_i). By 2), (G_n) satisfies (p,q)-estimates in $Z_p(G)$. □

It remains to sketch the proof of Theorem 6.9 b). The setup is we can regard $X^*\subseteq Z^*$, a reflexive space with a bimonotone FDD (E_i^*) so that in addition, using [JRZ], $\langle (E_i^*)\rangle \cap X^*$ is dense in X^*. So we have a quotient map $Q:Z\to X\subseteq W$, where by part a) W is a reflexive space with an FDD (F_i) satisfying C-$(p,1)$-estimates for some C. We choose $\bar\delta=(\delta_i)$, $\delta_i\downarrow 0$ rapidly and, by further blocking, may assume that any $\bar\delta$-skipped blocking of any blocking of (F_i) satisfies $2C$-$(p,1)$-estimates. We then may assume by further blocking using a lemma of Johnson and Zippin [JZ1, JZ2] that, given $\varepsilon_i\downarrow 0$, for all $i\le j$, $z\in S_{[E_n]_{n\in(i,j]}}$ that

$$\|P^F_{[1,i)} Qz\| < \varepsilon_i \quad \text{and} \quad \|P^F_{(j,\infty)}Qz\| < \varepsilon_j\ ,$$

in other words $Q([E_n]_{n\in(i,j]})$ is essentially contained in $\langle F_n\rangle_{n\in[i,j]}$.

We next renorm $\langle (E_i)\rangle$. We let $\tilde E_i$ be the quotient space of E_i determined by Q : for $z\in E_i$, $|||\tilde z||| = \|Qz\|$. For $\tilde z=\sum \tilde z_i\in \langle(\tilde E_i)\rangle$, $\tilde z_i\in \tilde E_i$ we set

$$|||\tilde z||| = \sup_{m\le n}\Big\|\sum_{i=m}^n Qz_i\Big\| \quad\text{and}\quad \tilde Z = \overline{(\langle(\tilde E_i)\rangle, |||\cdot|||)}\ .$$

One can prove

Lemma 6.10.

a) $(\tilde E_i)$ *is a bimonotone shrinking* FDD *for* $\tilde Z$.

b) *Setting* $\tilde Q(\sum \tilde z_i) = \sum Qz_i$, $\tilde Q$ *extends to a quotient map from* $\tilde Z$ *onto* X. *More precisely if* $x\in X$, $z\in Z$, $Qz=x$, $\|z\|=\|x\|$ *and* $z=\sum z_i$, $z_i\in E_i$ *then* $\tilde z=\sum \tilde z_i \in \tilde Z$, $|||\tilde z||| = \|z\|$ *and* $\tilde Q\tilde z = x$.

c) *If* $(\tilde z_i)$ *is a block sequence of* $(\tilde E_i)$ *in* $B_{\tilde Z}$ *and* $(\tilde Q\tilde z_i)$ *is a basic sequence with projection constant* $\bar K$ *and* $a=\inf_i\|\tilde Q\tilde z_i\|>0$ *then for all* (a_i),

$$\Big\|\sum a_i\tilde Q\tilde z_i\Big\| \le \Big|\Big|\Big|\sum a_i\tilde z_i\Big|\Big|\Big| \le \frac{3\bar K}{a}\Big\|\sum a_i\tilde Q\tilde z_i\Big\|\ .$$

The rest of the proof involves further blocking arguments and a number of estimates to accomplish the following. To produce $A<\infty$ and a blocking $\tilde H=(\tilde H_n)$ of $(\tilde E_n)$ so that: if $x\in S_X$ there exists $\tilde z=\sum \tilde z_n$, $\tilde z_n\in \tilde H_n$, so that if $(\tilde w_n)$ is any blocking of $(\tilde z_n)$ then $(\sum |||\tilde w_n|||^p)^{1/p}\le A$ and $\|\tilde Q\tilde z - x\|<1/2$. It follows that $\tilde Q:\tilde Z_p(\tilde H)\to X$ remains a quotient map and also $\tilde Z_p(\tilde H)$ is reflexive with 1-$(p,1)$-estimates for the FDD $\tilde H$. The idea is to form $\tilde H$ so that $(\tilde Q\tilde w_n)$ is essentially a skipped block sequence in W so behaves like an ℓ_p sum. As in the proof of part a) one needs to use a different breaking of the sum into $\sum \tilde z_n$ where the $\tilde z_n$'s and $\tilde w_n$'s overlap. □

A consequence of Theorem 6.8 is

Theorem 6.11. *There exists a reflexive space* X *which is universal for the class* $\{Y : Y$ *is uniformly convex*$\}$.

Definition. A space X is *universal* for a class $\mathcal C$ of Banach spaces if for all $Y\in\mathcal C$, $Y\hookrightarrow X$.

Bourgain proved that if X is universal for $\mathcal C=\{Y : Y$ is reflexive $\}$ then $C[0,1]\hookrightarrow X$. Prus [Pr] gave a partial result by proving that there exists a reflexive space X universal for the class

$$\{Y : Y \text{ has an FDD satisfying } (p,q)\text{-estimates for some } 1<q\le p<\infty\}\ .$$

Theorem 6.11 thus follows from Prus' result, Theorem 6.8 and the fact that every uniformly convex space satisfies (p,q)-tree estimates for some $1<q\le p<\infty$. Indeed James [J5] proved (see also [GG]) that if Y is uniformly convex there exists K and $1<q\le p<\infty$ so that if (y_i) is normalized 2-basic in Y then (y_i) satisfies K-(p,q)-estimates.

Remark. One way to deduce Prus' result for a fixed $1<q\le p<\infty$ is to use Schechtman's proof of Theorem 6.1 for FDD's (which he also gave). One can find a tree $(F_\alpha)_{\alpha\in T_\infty}$ of finite dimensional spaces with the nodes dense among all finite dimensional spaces and construct a norm on $\langle(F_\alpha)\rangle$ so that each branch satisfies 1-(p,q)-estimates and every Y with an FDD satisfying 1-(p,q)-estimates is equivalent to a branch. We then take $Z_{(p,q)}$ to be the completion of $\langle (F_\alpha)\rangle$ under

$$\|z\| = \sup\Big\{\Big(\sum_j \|P^F_{I_j} z\|^p\Big)^{1/p} : I_1, I_2, \ldots, \text{ are disjoint segments in } T_\infty\Big\}\ .$$

One checks that (F_α) is a 1-(p,q) FDD for $Z_{(p,q)}$ and let $X = (\sum Z_{(p_n,q_n)})_{\ell_2}$ for $q_n \downarrow 1$, $p_n \uparrow \infty$ (see [OS7] for details).

The following remains open

Problem. Let X be superreflexive (or uniformly convex). Does X embed into a superreflexive space (or uniformly convex space) with a basis or an FDD?

X is *superreflexive* if whenever Y is finitely representable in X then Y is reflexive. It turns out that being superreflexive is the same as being isomorphic to a uniformly convex space ([E3], [Pi]).

Our next topic will be spaces with X^* separable. As mentioned Zippin showed $X \hookrightarrow Z$, a space with a shrinking basis, but what more can be said? Well we need some more definitions. Szlenk [Szl] defined an ordinal index for spaces with separable dual which measures, in some sense, how big X^* is.

Definition. Let $\varepsilon > 0$. Let $K_0(X,\varepsilon) = B_{X^*}$ and for $\alpha < \omega_1$ let

$$K_{\alpha+1}(X,\varepsilon) = \{x^* \in K_\alpha(X,\varepsilon) : \exists\ (x_n^*) \subseteq K_\alpha(X,\varepsilon)$$

$$\text{with } \omega^* - \lim_{n\to\infty} x_n^* = x^* \text{ and } \varliminf_{n\to\infty} \|x_n^* - x^*\| \ge \varepsilon\}$$

If α is a limit ordinal,

$$K_\alpha(X,\varepsilon) = \bigcap_{\beta<\alpha} K_\alpha(X,\varepsilon)$$

$$S_Z(X,\varepsilon) = \inf\{\alpha : K_\alpha(X,\varepsilon) = \emptyset\}$$

or ω_1 otherwise

$$S_Z(X) = \sup\{S_Z(X,\varepsilon) : \varepsilon > 0\}\ .$$

Szlenk showed that if X^* is separable and $K_\alpha(X,\varepsilon) \ne \emptyset$ then $K_{\alpha+1}(X,\varepsilon)$ is a proper ω^*-closed subset of $K_\alpha(X,\varepsilon)$. Hence $S_Z(X) < \omega_1$ iff X^* is separable. The smallest possible value of $S_Z(X)$ is ω, and we shall discuss these spaces shortly. First we need a number of definitions.

Definition. Let $X \subseteq Z$, a space with a boundedly complete FDD (E_i) (and thus Z is naturally a dual space). X satisfies C-(p,q)-ω^* *tree estimates* if every ω^*-null tree in S_X admits branches (y_i) and (z_i) satisfying

$$C^{-1}\Big(\sum |a_i|^p\Big)^{1/p} \le \Big\|\sum a_i y_i\Big\| \text{ and } \Big\|\sum a_i z_i\Big\| \le C\Big(\sum |a_i|^q\Big)^{1/q}$$

for all scalars (a_i).

A ω^*-null tree is one whose nodes are ω^*-null sequences. As in Theorem 6.9 a) we can prove the following

Proposition 6.12. *Let $X \subseteq Z$ where Z has a boundedly complete* FDD *$E = (E_i)$ and B_X is ω^*-closed in Z. Let $1 \le p \le \infty$. If X satisfies $(p,1)$-ω^* tree estimates then for some blocking $F = (F_i)$ of E, X naturally embeds into $Z_p(F)$.*

Definition. X has the *ω^*-UKK* if for all $\varepsilon > 0$ there exits $\delta(\varepsilon) > 0$ so that if $(x_n^*) \subseteq S_{X^*}$ converges ω^* to x^* and $\underline{\lim}\, \|x_n^* - x^*\| \geq \varepsilon$ then $\|x^*\| \leq 1 - \delta(\varepsilon)$.

UKK stands for uniform Kadec-Klee. For example all the ℓ_p's are ω^*-UKK, $1 < p < \infty$, and in fact ω-UKK.

Indeed if $\|x_n\| = 1$, $x_n \xrightarrow{\omega} x$ in ℓ_p and $\underline{\lim}\, \|x_n - x\| \geq \varepsilon$ then $\|x\|^p + \underline{\lim}\, \|x_n - x\|^p \leq 1$ so $\|x\| \leq (1 - \varepsilon^p)^{1/p}$.

Definition. [JLPS] The *modulus of asymptotic uniform smoothness* $\bar\rho_X(t)$, $t > 0$ is

$$\bar\rho_X(t) = \sup_{\|x\|=1} \inf_{Y \in \operatorname{cof}(X)} \sup_{y \in tB_Y} \|x + y\| - 1\ .$$

The modulus of *asymptotic uniform convexity* $\bar\delta_X(t)$, $t > 0$ is

$$\bar\delta_X(t) = \inf_{\|x\|=1} \sup_{Y \in \operatorname{cof}(X)} \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|x + y\| - 1\ .$$

X is *asymptotically uniformly smooth* (a.u.s) if $\lim_{t \to 0^+} \bar\rho_X(t)/t = 0$.
X is *asymptotically uniformly convex* (a.u.c.) if for all $t > 0$, $\bar\delta_X(t) > 0$.

So for example ℓ_1 is a.u.c. and c_0 is a.u.s. but not vice versa.
X is *a.u.s. of power type* p if for some K

$$\bar\rho_X(t) \leq Kt^p \quad \text{for} \quad t > 0\ .$$

X is *a.u.c. of power type* p if for some K

$$\bar\delta_X(t) \geq Kt^p \quad \text{for} \quad t > 0\ .$$

If X is a.u.c. of power type p then $\exists\ c > 0$ such that for all $(e_i)_1^n \in \{X\}_n$, $\|\sum_1^n a_i e_i\| \geq c(\sum_1^n |a_i|^p)^{1/p}$ and the reverse inequality holds if X is a.u.s. of power type p.

We can also define $\bar\delta_X^*(t)$ and $\bar\rho_X^*(t)$ if X is a dual space using $\operatorname{cof}^*(X) = \{Y \subseteq X : Y$ is ω^*-closed finite co-dimensional subspace of $X\}$.

Lemma 6.13. a) *Let X^* be separable. If X is a.u.c. of power type p then X satisfies $(p,1)$-tree estimates.*

b) *Let X^* be separable. If X is a.u.s. of power type q then X satisfies (∞, q)-tree estimates.*

c) *Let $X = Y^*$. If X is ω^*-a.u.c. of power type p then X satisfies $(p,1)$-ω^* tree estimates.*

Theorem 6.14. *Let X^* be separable. The following are equivalent.*

1) $S_Z(X) = \omega$.
2) $\exists\ q > 1\ \exists\ K < \infty\ \forall\ n \forall\ (e_i)_1^n \in \{X\}_n\ \forall\ (a_i)_1^n \subseteq \mathbb{R}$, $\|\sum_1^n a_i e_i\| \leq K(\sum_1^n |a_i|^q)^{1/q}$.
3) $\exists\ q > 1$ *so that X satisfies (∞, q)-tree estimates.*
4) $\exists\ p < \infty$ *so that X^* satisfies $(p,1)$-ω^* tree estimates.*
5) $\exists\ p < \infty\ \exists\ Z$ *with a boundedly complete* FDD E *so that $X \hookrightarrow Z_p(E)$ as a ω^*-closed subspace.*
6) *X can be renormed to be a.u.s. of power type p for some $q > 1$.*
7) *X can be renormed to be a.u.s.*
8) *X can be renormed so that $\bar\rho_X(t) < t$ for some $t > 0$.*
9) *X can be renormed to be ω^*-UKK with $\delta(\varepsilon) \geq c\varepsilon^p$ for some $p < \infty$, $c > 0$.*
10) *X can be renormed to be ω^*-UKK.*
11) $\exists\ p < \infty$ *so that X can be renormed so that $\bar\delta_{X^*}^*(t) \geq Kt^p$ for some K.*
12) *X can be renormed to be ω^*-a.u.c.*

Some of these results were also obtained in [GKL]. We won't go into the details of the proof which involves a few more things than we have discussed (see [OS7]). The equivalence of 1) and 2) follows from a different interpretation of $S_Z(X)$ in terms of the index of certain ℓ_1^+-weakly null trees of which one consequence is:

$$S_Z(X) > \omega \ \text{ iff } \ \forall\ K > 1\ \forall\ n\ \exists\ (e_i)_1^n \in \{X\}_n$$

with $\|\sum_1^n a_i e_i\| \geq K \sum_1^n a_i$ for $(a_i)_1^n \subseteq [0,\infty)$, (see [AJO]). When this fails we obtain some upper ℓ_q estimate ([J5], [Jo3]).

Remark. We thus can also see that the universal reflexive space for the class of uniformly convex spaces of Theorem 6.11 is universal for

$$\mathcal{C}_\omega \equiv \{Y : Y \ \text{ is reflexive, } \ S_Z(Y) = \omega \ \text{ and } \ S_Z(Y^*) = \omega\}\ .$$

We discussed three notions of being asymptotic ℓ_p. The first was with respect to a basis which generalizes to an FDD (E_i). (E_i) is *K-asymptotic ℓ_p* if for all n and all normalized block sequences $(x_i)_1^n$ of $(E_i)_n^\infty$, $\|\sum_1^n a_i x_i\| \overset{K}{\sim} (\sum_1^n |a_i|^p)^{1/p}$. We also talked about every $(e_i)_1^n \in \{X\}_n$ being K-equivalent to the unit vector basis of ℓ_p^n and called X an asymptotic ℓ_p space. The third notion was in terms of $\{X\}_n^t$ when X had a basis (generalizing to an FDD) which coincided with X being asymptotic ℓ_p if the FDD is shrinking.

Our next theorem yields that every reflexive asymptotic ℓ_p space embeds into one with an asymptotic ℓ_p FDD.

We need to extend our notion of (p,q)-tree estimates and FDD's with (p,q)-estimates to more general sequences to prove the result.

Definition. Let (u_i) and (v_i) be normalized 1-unconditional bases for spaces U and V. X satisfies *C-(V,U)-tree estimates* if every weakly null tree in X admits branches (y_i) and (z_i) satisfying $C^{-1}\|\sum a_i v_i\| \leq \|\sum a_i y_i\|$ and $\|\sum a_i z_i\| \leq C\|\sum a_i u_i\|$ for $(a_i) \subseteq \mathbb{R}$.

An FDD (E_i) satisfies C-(V,U) *estimates* if for all normalized block sequences (x_i) of (E_i),

$$C^{-1}\|\sum a_i v_i\| \leq \|\sum a_i x_i\| \leq C\|\sum a_i u_i\|\ ,$$

for $(a_i) \subseteq \mathbb{R}$.

Clearly (u_i) dominates (v_i) if this holds and we need some more technical conditions to get a theorem.

Definition. Let (u_i) be a normalized 1-unconditional basis.
(WLS) There exists $d > 0$ so that for all m there exists $L = L(m) \geq m$ so that for all $k \leq m$

$$\Big\|\sum_{i=L+1}^\infty a_i u_{i-k}\Big\| \geq d\Big\|\sum_{i=L+1}^\infty a_i u_i\Big\|$$

for all (a_i).
(SRS) There exists $c > 0$ so that for all n

$$\Big\|\sum_{i=1}^\infty a_i u_{i+n}\| \geq c\Big\|\sum a_i u_i\Big\|\ .$$

(UB) (u_i) is dominated by every normalized block basis of (u_i).

(WLS) is for weak left shift and (SRS) is for strong right shift, and (UB) is for upper block (domination).

Let (v_i) be another 1-unconditional normalized basis.

($*$) Every (u_{n_i}) dominates every normalized block basis of (v_i) and every normalized block basis of (u_i) dominates every (v_{n_i}).

Theorem 6.15. [OSZ2] *Let V and U be reflexive spaces with normalized 1-unconditional bases (v_i) and (u_i). Assume that (v_i) and (u_i^*) satisfy* (WLS), (SRS) *and* (UB) *and that* ($*$) *holds for the pair $((v_i),(u_i^*))$.*

Let X be reflexive and satisfy (V,U)-tree estimates. Then X embeds into a reflexive space Z with an FDD satisfying (V,U)-estimates.

The proof follows the general guidelines of the proof of the (p,q)-estimates Theorem 6.8 but there are technical problems to overcome which lengthen the arguments substantially.

So how does this apply to asymptotic ℓ_p spaces? For $0<\gamma<1$, $1\le p<\infty$ we let $T_{p,\gamma}$ be the Tsirelson space given by the implicit norm on c_{00},

$$\|x\| = \|x\|_\infty \vee \sup\left\{\gamma\Big(\sum_1^n \|P_{A_i}x\|^p\Big)^{1/p} : n\ge 2\ ,\ n\le A_1<\cdots<A_n\right\} .$$

The unit vector basis is 1-unconditional and asymptotic ℓ_p. $T_{p,\gamma}$ is reflexive.

Definition. Let $1\le q\le p\le\infty$. We will say that X satisfies *asymptotic C-(p,q)-tree estimates* if for all $(e_i)_1^n\in\{X\}_n$, for all $(a_i)_1^n\subseteq\mathbb{R}$,

$$C^{-1}\Big(\sum_1^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_1^n a_ie_i\Big\| \le C\Big(\sum_1^n |a_i|^q\Big)^{1/p} .$$

An FDD (E_i) satisfies *asymptotic C-(p,q)-estimates* if for all n and all normalized block bases $(x_i)_1^n$ of $\langle E_i\rangle_{i\ge n}$,

$$C^{-1}\Big(\sum_1^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_1^n a_ix_i\Big\| \le C\Big(\sum_1^n |a_i|^q\Big)^{1/q} .$$

One can prove that a reflexive X satisfies asymptotic (p,q)-tree estimates iff X satisfies $(T_{p,\gamma},T_{q',\gamma})$-tree estimates for some $0<\gamma<1/4$.

We also can show that for $0<\gamma<1/4$, the spaces $(U,V)=(T_{p,\gamma},(T_{q',\gamma})^*)$ satisfy the conditions of Theorem 6.15.

Theorem 6.16. [OSZ2] *Let $1\le q\le p\le\infty$. Let X be reflexive. The following are equivalent.*

a) *X satisfies asymptotic (p,q)-tree estimates*
b) *X embeds into a reflexive space Z with an* FDD *satisfying asymptotic (p,q)-estimates*
c) *X is a quotient of a reflexive space Z with an* FDD *satisfying asymptotic (p,q)-estimates*
d) *X^* satisfies asymptotic (q',p')-tree estimates.*

With some of the usual universal space construction tricks we obtain

Corollary 2. *Given $K<\infty$ and $1<p<\infty$, there exists a universal reflexive asymptotic ℓ_p space for the class*

$$C=\{Y : Y \ \text{ is reflexive and } K\text{-asymptotic } \ \ell_p\} .$$

The case $p = 1$ or ∞ must be excluded by Bourgain's index theory. Indeed if X contains isomorphs of the spaces $\{T(S_\alpha, 1/2) : \alpha < \omega_1\}$ then uncountably many of these uniformly embed into X and we obtain $\ell_1 \hookrightarrow X$. Similarly $c_0 \hookrightarrow X$ if the spaces $T(S_\alpha, 1/2)^*$ embed into X for $\alpha < \omega_1$. Also the K is needed.

Our next object is to extend our proof of the existence of a reflexive space universal for C_ω to one universal for C_α, $\alpha < \omega_1$.

$$C_\alpha = \{Y : Y \text{ is reflexive, } S_Z(Y) \le \alpha \ , \ S_Z(Y^*) \le \alpha\} \ .$$

In this setting if $\alpha > \omega$ we cannot hope to obtain (p,q)-tree estimates so we need to replace them with something else, a bit more complex, and which leads to a new game [OSZ1]. As usual we need definitions of estimates for an FDD and also a coordinate free version.

Definition. Let Z be a space with an FDD (E_n) and let V have a normalized 1-unconditional basis (v_i). (E_n) satisfies *subsequential C-V-lower estimates* if every normalized block sequence (z_i) of (E_n) satisfies for all (a_i)

$$\Big\| \sum a_i v_{m_i} \Big\| \le C \Big\| \sum a_i z_i \Big\|$$

whenever $m_i = \min \operatorname{supp}_E z_i$.

We can similarly define *subsequential C-V-upper estimates* and, jointly, *subsequential C-(V,U)-estimates* if U is another space with a normalized 1-unconditional basis (u_i).

So the estimates in this case depend upon the support of z_n.

Recall $T_\infty = \bigcup_{n\in\mathbb{N}} T_n$. We define $T_\infty^{\text{even}} = \bigcup_{n=1}^\infty T_{2n}$.

Definition. An *even tree* in X is $(x_\alpha)_{\alpha\in T_\infty^{\text{even}}} \subseteq X$. *Nodes* in this tree are sequences $(x_{(\alpha,n)})_{n>n_{2\ell-1}}$ where $\alpha = (n_1, n_2, \dots, n_{2\ell-1})$. A *branch* of the tree is $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$. The *tree* is *normalized* if $x_\alpha \in S_X$ for all α, and it is *weakly null* if all nodes are weakly null.

Definition. Let V have a normalized 1-unconditional basis (v_i). X satisfies *subsequential C-V-lower tree estimates* if every normalized weakly null even tree in X admits a branch $(x_{(n_1,\dots,n_{2i})})_{i=1}^\infty$ which C-dominates $(v_{n_{2i-1}})$, i.e.,

$$\Big\| \sum_i a_i v_{n_{2i-1}} \Big\| \le C \Big\| \sum_i a_i x_{n_1,\dots,n_{2i})} \Big\| \ .$$

This extends as before to *subsequential C-V-upper tree estimates* and *C-(V,U)-tree estimates*.

The object is to embed a reflexive X satisfying subsequential C-(V,U)-tree estimates into a reflexive Z with an FDD satisfying (V,U)-estimates. As one expects certain conditions are required on U and V.

Definition. Let (v_i) be normalized 1-unconditional. (v_i) is *C-block stable* (also called *block determined* by H. Rosenthal) if for all normalized block bases (x_i) and (y_i) of (v_i) with

$$\max(\operatorname{supp}(x_i) \cup \operatorname{supp}(y_i)) < \min(\operatorname{supp}(x_{i+1}) \cup \operatorname{supp}(y_{i+1}))$$

for all i, $(x_i) \overset{C}{\sim} (y_i)$.

(v_i) is *C-right dominant* if for all $m_1 < m_2 < \cdots$, $n_1 < n_2 < \cdots$ with $m_i \le n_i$ for all i, (v_{m_i}) is C-dominated by (v_{n_i}). If we change the latter to (v_{n_i}) is C-dominated by (v_{m_i}) we say (v_i) is *C-left dominant*.

Normalized 1-unconditional bases (u_i) and (v_i) are *regular* if $U = [(u_i)]$, $V = [(v_i)]$ are both reflexive, (v_i) is left dominant, (u_i) is right dominant and (u_i) dominates (v_i). If U and V are regular we set

$$\{\mathcal{A}_{V,U}(C) = \{Y : Y \ \text{ is reflexive and satisfies subsequential } C\text{-}(V,U)\text{-tree estimates}\} \ .$$

The arguments that every $Y \in \mathcal{A}_{V,U}(C)$ embeds into a reflexive Z with an FDD satisfying subsequential (V,U)-estimates follows the general pattern as that for (p,q)-estimates or for (V,U)-estimates given previously, with some additional difficulties. One first establishes embedding and quotient results for the lower estimate and uses duality arguments to also get the upper one. The role of $Z_P(E)$ or $Z_V(E)$ is played by $Z^V(E)$. $Z^V(E)$ is the completion of $\langle (E_i) \rangle$ under

$$\|z\|_{Z^V} = \max_{\substack{k\in\mathbb{N} \\ 1\le n_0<n_1<\cdots<n_k}} \Big\| \sum_{j=1}^{k} \|P^E_{[n_{j-1},n_j)} z\| v_{n_{j-1}} \Big\|_V \ .$$

It is also established that

Theorem 6.17. *Given (V,U) regular, $C < \infty$ there exists $\bar{C}$ and a reflexive Z with an* FDD *satisfying subsequential $\bar{C}$-(V,U)-estimates that is universal for $\mathcal{A}_{V,U}(C)$.*

$\bar{C}$ depends on C and also the constants involved in the regularity of (V,U).

So how does this apply to our universal reflexive space for $\mathcal{C}_\alpha$ problem? Well it turns out that if X is reflexive it satisfies for some $c \in (0,1/4)$ subsequential upper $T_{\alpha,c}$-tree estimates where $\alpha = \alpha(S_z(X))$. Here $T_{\alpha,c} = T(S_\alpha, c)$ is the higher order Tsirelson space. So by duality we also get the subsequential lower $T^*_{\alpha,c}$-tree estimates. Now $S_Z(T_{\alpha,c}) = \omega^{\alpha\cdot\omega}$ and it can be shown that $(T^*_{\alpha,c}, T_{\alpha,c})$ are regular. To be more precise we have from [OSZ3]

Theorem 6.18. *Let $\alpha < \omega_1$ and let X be reflexive. The following are equivalent.*

i) $X \in \mathcal{C}_{\omega^{\alpha\cdot\omega}}$
ii) *X satisfies subsequential $(T^*_{\alpha,c}, T_{\alpha,c})$-tree estimates for some $0 < c < 1/4$.*
iii) *X embeds into a reflexive space Z with an* FDD *satisfying subsequential $(T^*_{\alpha,c}, T_{\alpha,c})$ estimates, for some $0 < c < 1/4$.*

Theorem 6.19. *For every $\alpha < \omega_1$ there is a separable reflexive space Z with an* FDD *which satisfies subsequential $(T^*_{\alpha,c}, T_{\alpha,c})$ estimates for some $c \in (0,1/4)$, hence $Z \in \mathcal{C}_{\omega^{\alpha\cdot\omega}}$, and Z is universal for $\mathcal{C}_{\omega^\alpha}$.*

In fact one can get $Z \in \mathcal{C}_{\omega^{\alpha\cdot\omega+1}}$ with an FDD which is universal for $\mathcal{C}_{\omega^{\alpha\cdot\omega}}$.

Other results exist using weakly null trees to characterize embeddings and we next present some of these.

Definition. X has the *bounded tree property* if for all normalized weakly null trees in X some branch (x_i) satisfies, $\sup_n \|\sum_1^n x_i\| < \infty$.

As before in this case X must have the *C-bounded tree property* for some $C < \infty$,

$$\sup_n \|\sum_1^n x_i\| \le C \ .$$

Theorem 6.20. [Ka] *Assume $\ell_1 \not\hookrightarrow X$ and X has the bounded tree property. Then $X \hookrightarrow c_0$.*

The stronger result than just $X \overset{2}{\hookrightarrow} (\sum E_n)_{c_0}$ is not surprising. Indeed $(\sum E_n)_{c_0} \hookrightarrow c_0$ since every finite dimensional $E_n \overset{2}{\hookrightarrow} c_0$. Now X^* must be separable or else it contains a K-ℓ_1^+-weakly null tree; all branches satisfy $\|\sum a_i x_i\| \geq \frac{1}{K}\sum a_i$ for $(a_i) \subseteq [0,\infty)$, and this contradicts the bounded tree property, (see e.g. [AJO]). Thus we may begin with $X \subseteq Z$, a space with a bimonotone shrinking basis (Theorem 6.3).

One can produce a blocking (E_i) of the basis for Z and $\bar{\delta}$ so that every $\bar{\delta}$-skipped block sequence (x_i) in X satisfies $\|\sum_1^n x_i\| \leq 2C$ for all n. Hence $\|\sum_1^n \pm x_i\| \leq 2C$ as well so $\|\sum_1^n a_i x_i\| \leq 2C \max|a_i|$. So X satisfies (∞,∞)-tree estimates. So X^* satisfies $(1,1)$-ω^*-tree estimates and embeds ω^*-closed into $Z_1^*(F_i^*) = (\sum F_i^*)_{\ell_1}$ for some blocking (F_i) of (E_i). So from basic functional analysis X is a quotient of $(\sum F_i)_{c_0}$ so X is a subspace of a quotient of c_0 so $X \hookrightarrow c_0$ [JZ2].

Next we have a very recent beautiful result of Johnson and Zheng.

Theorem 6.21. [JZh] *Let X be reflexive and assume that every weakly null tree in S_X admits an unconditional branch. Then X embeds into a reflexive space with an unconditional basis.*

We will sketch the proof. The end game is to show X embeds into a reflexive Z, a space with an unconditional FDD, and then use [LT1] that Z embeds into a reflexive space with an unconditional basis.

As is usual if every weakly null tree in S_X admits an unconditional branch then it admits a C-unconditional branch for some absolute $C < \infty$. We will say then that X has the (UTP), *unconditional tree property*. The steps in the proof are

1) If Y is reflexive with the (UTP) and X is a quotient of Y then X has the (UTP). This takes some work.
2) If $X_1 \oplus X_2 \subseteq X$ then naturally $X \hookrightarrow X/X_1 \oplus X/X_2$. This is easy.
3) [JR] If X^* is separable there exists $X_1 \subseteq X$ with a shrinking FDD (E_n) so that for all $(n_k) \in [\mathbb{N}]^\omega$, $X/[(E_{n_k})]$ has a shrinking FDD.
4) [OS7] Every FDD in a reflexive X with the (UTP) can be blocked into (E_n), an USB FDD, or unconditional skipped blocking FDD. This means that for some C if (x_n) is a skipped block sequence of (E_n) then (x_n) is C-unconditional.
5) For X reflexive the following are equivalent
   a) X has the (UTP)
   b) $X \hookrightarrow Z$, reflexive with USB FDD
   c) X^* has UTP

The new new step here is a) $\Rightarrow$ b). We use 3) to get $X \supseteq X_1 = [(E_n)]$ which can be taken to be USB FDD by 4). Set $Y_1 = [(E_{4n})]$, $Y_2 = [(E_{4n+2})]$ and note that X/Y_i has USB FDD. Now $Y_1 \oplus Y_2 \subseteq X$ so by 2) $X \hookrightarrow X/Y_1 \oplus X/Y_2$ which has USB FDD by 4).

6) X reflexive with the (UTP) inplies X embeds into a reflexive Z with an unconditional FDD.

By 5) we can assume X has an FDD (E_n) which by 4) can be assumed to be USB FDD. Set $Y_1 = [(E_{4n})]$, $Y_2 = [(E_{4n+2})]$ as before. Then $X \hookrightarrow X/Y_1 \oplus X/Y_2$ and $(X/Y_1)^* = Y_1^\perp = [(E_{4n-3}^* \oplus E_{4n-2}^* \oplus E_{4n-1}^*)_n]$ which is an unconditional FDD.

Johnson and Zheng go on to prove that if X is reflexive and is a quotient of a space with a shrinking unconditional basis, it has the (UTP), hence also embeds into a reflexive space with an unconditional basis.

B. Zheng [Zh1] obtained factorization theorem analogues of Theorem 6.4 and 6.21.

Definition. Let $1 \le p < \infty$, $C > 0$ and $T : X \to Y$ a bounded linear operator. T satisfies an *upper C-p-tree estimate* if for all normalized weakly null trees in X there exists a branch (x_i) satisfying

$$\|T(\sum a_i x_i)\| \le C(\sum |a_i|^p)^{1/p} \text{ for all scalars } (a_i) .$$

T satisfies an upper C-∞-tree estimate if every such tree admits a branch (x_i) with

$$\sup_n \|T(\sum_1^n x_i)\| \le C .$$

Theorem 6.22. [Zh1] *Let $T : X \to Y$ satisfy an upper C-p-tree estimate, $1 < p \le \infty$.*

a) *If X has a shrinking* FDD(E_n) *satisfying $(1,p)$-estimates then T factors through $(\sum F_n)_p$ for some blocking (F_n) of (E_n), i.e., there exist operators $R : X \to (\sum F_n)$ and $S : (\sum F_n)_p \to X$ with $T = SR$.*
b) *If $X = L_p$, $2 < p < \infty$, then T factors through ℓ_p.*

B. Zheng also proved the following results.

Theorem 6.23. [Zh2] *Let $1 < q < p < \infty$, $q \le r \le p$. Let X be a reflexive space satisfying $(q,1)$-tree estimates. Let $T : X \to Y$ be a bounded linear operator satisfying upper C-p tree estimates. Then T factors through a subspace of $(\sum F_n)_r$ for some sequence (F_n) of finite dimensional spaces.*

As we mentioned, if $\ell_1 \not\hookrightarrow X$ then X^* is not separable iff for all $K > 1$ there exists a weakly null tree in S_X so that all branches (x_i) satisfy $\|\sum_1^\infty a_i x_i\| \ge \frac{1}{K}\sum_1^\infty a_i$ whenever $(a_i) \subseteq [0,\infty)$. The separability of X^* can also be characterized in terms of ω^*-null trees in X^*.

Theorem 6.24. [DF] *X^* is separable iff every ω^*-null tree in S_{X^*} admits a boundedly complete branch.*

They also characterized the PCP.

Definition. X has the *PCP* (point of continuity property) if for all weakly closed bounded sets $A \subseteq X$ the identity map $i : (A, \omega) \to (A, \|\cdot\|)$ has a point of continuity.

Theorem 6.25. [DF] *Let X^* be separable. Then X has the* PCP *iff every weakly null tree in S_X admits a boundedly complete branch.*

We must also note that powerful set theoretic techniques have been used to construct universal spaces for certain classes $\mathcal{C}$. Bossard [Bos1, Bos2] considered (in these definitions, finite dimensional spaces are allowed)

$$\text{Sub} = \{X \subseteq C[0,1]\} \text{ and } \text{Sub}(X) = \{Y : Y \subseteq X\} .$$

On Sub, one considers the σ-algebra $\mathcal{F}[C[0,1]]$ generated by $\{X \in S\beta : X \cap U \ne \emptyset\}$ where U ranges over all open subsets of $C[0,1]$. This is called the Effros-Borel structure. Bossard proved

Theorem 6.26. [Bos1, Bos2] *If $\mathcal{A} \subseteq$ Sub is analytic and contains all reflexive spaces then $\mathcal{A}$ contains an isomorph of $C[0,1]$.*

Definition. [ADo] A class $\mathcal{C} \subseteq$ Sub is *strongly bounded* if for all analytic $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{C}$ contains a universal element for $\mathcal{A}$.

Theorem 6.27. [ADo], [DoF] *The class of separable reflexive spaces and the class of* $SD = \{X : X^* \text{ is separable}\}$ *are both strongly bounded.*

Theorem 6.28. [DoF] *Given* $\alpha < \omega_1$ *there exists* $X_\alpha \in SD$ *universal for* $\{X : S_Z(X) \leq \alpha\}$*.*

From Theorem 6.19 it follows that (communication of Dodos) $\mathcal{C}_{\omega^{\alpha\cdot\omega}}$ is analytic but this is, thus far, unobtainable from set theoretic arguments.

7. Small subspaces of L_p

In this section we discuss applications of the infinite asymptotic game/weakly null trees scenario (see e.g., [OS7]) to subspaces of $L_p = L_p[0,1]$. Our interest will be mainly the case $2 < p < \infty$.

We recall some of the basic structure of L_p (e.g., see [AO2]). The Haar basis (h_i) is a monotone basis for L_p $(1 \le p < \infty)$ which is K_p-unconditional iff $1 < p < \infty$;

$$\Big\|\sum a_i h_i\Big\| \le K_p \Big\|\sum \pm a_i h_i\Big\|$$

for all signs $\pm$ and all reals (a_i).

L_p contains the following "small" subspaces

- ℓ_p (isometrically): If $(x_i) \subseteq S_{L_p}$ are disjointly supported then

$$\Big\|\sum a_i x_i\Big\| = \Big(\int |\sum a_i x_i(t)|^p\,dt\Big)^{1/p} = \Big(\sum \int |a_i|^p |x_i(t)|^p\,dt\Big)^{1/p} = \Big(\sum |a_i|^p\Big)^{1/p}.$$

- ℓ_2 (isomorphically) via the Rademacher functions (r_n). (r_n) are ± 1 valued independent random variables of mean 0.

Khintchin's inequality: For $2 < p < \infty$,

$$\Big(\sum |a_n|^2\Big)^{1/2} = \Big\|\sum a_n r_n\Big\|_2 \le \Big\|\sum a_n r_n\Big\|_p \le B_p \Big(\sum |a_n|^2\Big)^{1/2}.$$

- ℓ_2 (isometrically) via a sequence of symmetric Gaussian independent random variables in S_{L_p}
- $(\ell_2 \oplus \ell_p)_p$ (isometrically)
- $(\sum \ell_2)_p \equiv \{(x_i) : x_i \in \ell_2 \text{ for all } i \text{ and } \|(x_i)\| = (\sum \|x_i\|_2^p)^{1/p} < \infty\}$ (isometrically)

Our topic will be to characterize when $X \subseteq L_p$, $2 < p < \infty$, embeds isomorphically into or contains isomorphically one of the four spaces ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$ or $(\sum \ell_2)_p$. We begin with some known results.

Proposition 7.1. *Let $2 < p < \infty$ and let $(x_i) \subseteq S_{L_p}$ be λ-unconditional. Then for all $(a_n) \subseteq \mathbb{R}$,*

$$\lambda^{-1}\Big(\sum |a_n|^p\Big)^{1/p} \le \Big\|\sum a_n x_n\Big\|_p \le \lambda B_p \Big(\sum |a_n|^2\Big)^{1/2}.$$

Proof. For $t \in [0,1]$,

$$\Big\|\sum a_n x_n\Big\|_p \le \lambda \Big\|\sum a_n x_n r_n(t)\Big\|_p$$

and so

$$\begin{aligned}
\Big\|\sum a_n x_n\Big\|_p^p &\le \lambda^p \int_0^1 \Big\|\sum a_n x_n r_n(t)\Big\|_p^p\,dt \\
&\overset{\text{(Fubini)}}{=} \lambda^p \int_0^1 \int_0^1 \Big|\sum a_n x_n(s) r_n(t)\Big|^p\,dt\,ds \\
&\le (\lambda B_p)^p \int_0^1 \Big(\sum a_n^2 x_n(s)^2\Big)^{p/2}\,ds \\
&\le (\lambda B_p)^p \Big(\sum \|a_n^2 x_n^2\|_{p/2}\Big)^{p/2}
\end{aligned}$$

(by the triangle inequality in $L_{p/2}$)

$$= (\lambda B_p)^p \Big(\sum |a_n|^2\Big)^{p/2} .$$

This gives the upper ℓ_2-estimate.

Similarly,

$$\begin{aligned}\lambda^p \Big\|\sum a_n x_n\Big\|^p &\geq \int_0^1 \Big(\sum a_n^2 x_n^2(s)\Big)^{p/2}\,ds \\ &\geq \int_0^1 \sum |a_n|^p |x_n(s)|^p\,ds = \sum |a_n|^p\end{aligned}$$

(using $\|\cdot\|_{\ell_p} \leq \|\cdot\|_{\ell_2}$). □

So unconditional basic sequences in L_p are trapped between the ℓ_p and ℓ_2 norms.

Theorem 7.2 (Kadets and Pełczyński [KP]). *Let $X \subseteq L_p$, $2 < p < \infty$. Then $X \sim \ell_2$ iff $\|\cdot\|_2 \sim \|\cdot\|_p$ on X; i.e., for some C, $\|x\|_2 \leq \|x\|_p \leq C\|x\|_2$ for all $x \in X$. Moreover there is a projection $P : L_p \to X$.*

The direction requiring proof is if $X \sim \ell_2$ then $\|\cdot\|_2 \sim \|\cdot\|_p$ on X. If not we can find $(x_i) \subseteq S_X$, $x_i \overset{\omega}{\to} 0$, so that for all $\varepsilon > 0$, $\lim_n m[|x_n| \geq \varepsilon] = 0$, where m is Lebesgue measure on $[0,1]$. From this we can construct a subsequence (x_{n_i}) and disjointly supported $(f_i) \subseteq S_{L_p}$ with $\lim_i \|x_{n_i} - f_i\| = 0$. Hence by a perturbation argument a subsequence of (x_i) is equivalent to the unit vector basis of ℓ_p which contradicts $X \sim \ell_2$.

The projection onto $X \overset{C}{\sim} \ell_2$ is given by the orthogonal projection $P : L_2 \to X$ acting on L_p. For $y \in L_p$,

$$\|Py\|_p \leq C\|Py\|_2 \leq C\|y\|_2 \leq C\|y\|_p .$$

□

Remarks. The proof yields that if $X \subseteq L_p$, $2 < p < \infty$, and $X \not\sim \ell_2$ then for all $\varepsilon > 0$, $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$.

Pełczyński and Rosenthal [PR] proved that if $d(X,\ell_2) \leq K$ then X is $C(K)$-complemented in L_p via a change of density argument.

Theorem 7.3. [JO2] *Let $2 < p < \infty$, $X \subseteq L_p$. Then $X \hookrightarrow \ell_p \Leftrightarrow \ell_2 \not\hookrightarrow X$. (*[KW] *If $\ell_2 \not\hookrightarrow X$ then for all $\varepsilon > 0$, $X \overset{1+\varepsilon}{\hookrightarrow} \ell_p$.)*

The scheme of the argument is to show if $\ell_2 \not\hookrightarrow X$ then there is a blocking (H_n) of the Haar basis into an FDD so that $X \hookrightarrow (\sum H_n)_p$ in a natural way; $x = \sum x_n$, $x_n \in H_n \to (x_n) \in (\sum H_n)_p$. By [Pe1], $(\sum H_n)_p \sim \ell_p$ since it is complemented in ℓ_p.

Let's digress for a moment to discuss subspaces of L_p $(1 < p < 2)$. The situation is more complemented here: $L_q \overset{1}{\hookrightarrow} L_p$ if $p \leq q \leq 2$. Johnson characterized when $X \subseteq L_p$ $(1 < p < 2)$ embeds into ℓ_p.

Theorem 7.4. [Jo2] *Let $X \subseteq L_p$, $1 < p < 2$. Then $X \hookrightarrow \ell_p$ if there exists $K < \infty$ so that for all weakly null $(x_i) \subseteq S_X$ some subsequence is K-equivalent to the unit vector basis of ℓ_p.*

These results were unified using the infinite asymptotic game/weakly null trees machinery.

Theorem 7.5. *Let $X \subseteq L_p$, $1 < p < \infty$. Then $X \hookrightarrow \ell_p$ iff every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p.*

A weakly null tree in S_X is $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ where

$$T_\infty = \{(n_1, \ldots, n_k) : k \in \mathbb{N}\ ,\ n_1 < \cdots < n_k \text{ are in } \mathbb{N}\}\ .$$

A *node* in T_∞ is all $(x_{(\alpha,n)})_{n > n_k}$ where $\alpha = (n_1, \ldots, n_k)$ or $\alpha = \emptyset$. The tree is *weakly null* means each node is a weakly null sequence. A *branch* is (x_i) given by $x_i = x_{(n_1,\ldots,n_i)}$ for some subsequence (n_i) of $\mathbb{N}$.

The proof is the same as that in [OS6] used to characterize when a reflexive space $X \hookrightarrow (\sum F_n)_p$. It yields a blocking (H_n) of (h_n) so that $X \hookrightarrow (\sum H_n)_p$.

Returning to $X \subseteq L_p$ $(2 < p < \infty)$ we have seen that one of these holds:

- $X \sim \ell_2$
- $X \hookrightarrow \ell_p$
- $\ell_p \oplus \ell_2 \hookrightarrow X$

Our goal will be to characterize when $X \hookrightarrow \ell_p \oplus \ell_2$ and if not to then show that $(\sum \ell_2)_p \hookrightarrow X$. First we recall one more old result.

Theorem 7.6. [JO3] *Let $X \subseteq L_p$, $2 < p < \infty$. Assume there exists $Y \subseteq \ell_p \oplus \ell_2$ and a quotient (onto) map $Q : Y \to X$. Then $X \hookrightarrow \ell_p \oplus \ell_2$.*

This is an answer, of a sort, to when $X \hookrightarrow \ell_p \oplus \ell_2$ but it is not an intrinsic characterization. The proof however provides a clue as to how to find one. The isomorphism $X \hookrightarrow \ell_p \oplus \ell_2$ is given by a blocking (H_n) of (h_i) so that X naturally embeds into

$$\Big(\sum H_n\Big)_p \oplus \Big(\sum (H_n, \|\cdot\|_2)\Big)_2 \sim \ell_p \oplus \ell_2\ .$$

Before proceeding we recall some more inequalities.

Theorem 7.7. [Ro5] *Let $2 < p < \infty$. There exists $K_p < \infty$ so that if (x_i) is a normalized mean zero sequence of independent random variables in L_p then for all $(a_i) \subseteq \mathbb{R}$,*

$$\Big\| \sum a_i x_i \Big\|_p \overset{K_p}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum |a_i|^2 \|x_i\|_2^2\Big)^{1/2}\ .$$

Note that in this case $[(x_i)] \hookrightarrow \ell_p \oplus \ell_2$ via the embedding

$$\sum a_i x_i \longmapsto \Big((a_i)_i, (a_i \|x_i\|_2)_i\Big) \in \ell_p \oplus \ell_2\ .$$

The next result generalizes this to martigale difference sequences, e.g., block bases of (h_i).

Theorem 7.8. [Bu2], [BDG] *Let $2 < p < \infty$. There exists $C_p < \infty$ so that if (z_i) is a martingale difference sequence in L_p with respect to the sequence of σ-algebras $(\mathcal{F}_n)$, then*

$$\Big\| \sum z_i \Big\|_p \overset{C_p}{\sim} \Big\| \sum \|z_i\|_p^p \Big)^{1/p} \vee \Big\| \Big(\sum \mathbb{E}_{\mathcal{F}_i}(z_{i+1}^2)\Big)^{1/2} \Big\|_?\ .$$

Now suppose that $(x_i) \subseteq S_{L_p}$ is weakly null. Passing to a subsequence we obtain (y_i) which, by perturbing, we may assume is a block basis of (h_i). Passing to a further subsequence we may assume $\varepsilon \equiv \lim_i \|y_i\|_2$ exists. If $\varepsilon = 0$ a subsequence of (y_i) is equivalent to the unit vector basis of ℓ_p by the [KP] arguments. Otherwise we have (essentially)

$$\varepsilon \Big(\sum |a_i|^2\Big)^{1/2} = \Big\| \sum a_i y_i \Big\|_2 \le \Big\| \sum a_i y_i \Big\|_p \le C(p) \Big(\sum |a_i|^2\Big)^{1/2}\ ,$$

using the fundamental inequality, Proposition 7.1.

Johnson, Maurey, Schechtman and Tzafriri obtained a stronger version of this dichotomy using Theorem 7.8.

Theorem 7.9. [JMST] *Let $2 < p < \infty$. There exists $D_p < \infty$ with the following property. Every normalized weakly null sequence in L_p admits a subsequence (x_i) satisfying, for some $w \in [0,1]$ and all $(a_i) \subseteq \mathbb{R}$,*

$$\Big\| \sum a_i x_i \Big\|_p \overset{D_p}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee w \Big(\sum |a_i|^2 \Big)^{1/2} .$$

We are now ready for an intrinsic characterization of when $X \subseteq L_p$ embeds into $\ell_p \oplus \ell_2$.

Theorem 7.10. [HOS] *Let $X \subseteq L_p$, $2 < p < \infty$. The following are equivalent.*

a) $X \hookrightarrow \ell_p \oplus \ell_2$

b) *Every weakly null tree in S_X admits a branch (x_i) satisfying for some K and all (a_i)*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2$$

c) *Every weakly null tree in S_X admits a branch (x_i) satisfying for some K and $(w_i) \subseteq [0,1]$ and all (a_i),*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum |a_i|^2 w_i^2 \Big)^{1/2} .$$

d) *There exists K so that every weakly null sequence in S_X admits a subsequence (x_i) satisfying the condition in b):*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2 .$$

Condition c) just says that every weakly null tree in S_X admits a branch equivalent to a block basis of the natural basis for $\ell_p \oplus \ell_2$.

Condition d) yields that we can find a subsequence (x_i) with $\| \sum a_i x_i \| \overset{K}{\sim} (\sum |a_i|^p)^{1/p} \vee (\sum a_i^2 \|x_i\|_2^2)^{1/p}$.

Conditions b) and c) do not require K to be universal but the “all weakly null trees...” hypothesis yields this.

Condition d) is an anomaly in that usually “every sequence has a subsequence...” is a vastly different condition than “every tree admits a branch...”. Here the special nature of L_p is playing a role.

The embedding of X into $\ell_p \oplus \ell_2$ will follow the clue from Theorem 7.6 by producing a blocking (H_n) of (h_i) and embedding X naturally into $(\sum H_n)_p \oplus (\sum (H_n, \|\cdot\|_2))_2$.

Now b) $\Rightarrow$ a) by using the infinite asymptotic game machinery of [OS7]. We won't go into that (again) here but we will note the argument proves the following

Theorem 7.11. *Let X and Y be Banach spaces with X reflexive. Let V be a space with a 1-subsymmetric normalized basis (v_i) and let $T : X \to Y$ be a bounded linear operator. Assume that for some C every normalized weakly null tree in X admits a branch (x_n) satisfying:*

$$\Big\| \sum a_n x_n \Big\|_X \overset{C}{\sim} \Big\| \sum a_n v_n \Big\|_V \vee \Big\| T\Big(\sum a_n x_n \Big) \Big\|_Y .$$

Then if $X \subseteq Z$, a reflexive space with an FDD(E_i), *there exists a blocking (G_i) of (E_i) so that X naturally embeds into $(\sum G_i)_V \oplus Y$.*

This is applied to $V = \ell_p$, $Z = L_p$ and $Y = L_2$ where $T : X \to L_2$ is the identity map.

So we obtain b) $\Rightarrow$ a) and clearly a) $\Rightarrow$ c). To see c) $\Rightarrow$ b) we begin with a weakly null tree in S_X and choose a branch (x_i) satisfying the c) condition:

$$\Big\|\sum a_i x_i\Big\| \overset{K}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum |a_i|^2 |w_i|^2\Big)^{1/2} .$$

We want to say that for some K',

$$\Big\|\sum a_i x_i\Big\| \overset{K'}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big\|\sum a_i x_i\Big\|_2 .$$

(We have $\underset{K'}{\geq}$ by the fundamental inequality.)

If this fails we can find a block basis (y_n) of (x_n),

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i \ , \text{ with } \sum_{i=k_{n-1}+1}^{k_n} w_i^2 a_i^2 = 1 \text{ and } \Big(\sum_{i=k_{n-1}+1}^{k_n} |a_i|^p\Big)^{1/p} \vee \|y_n\|_2 < 2^{-n} .$$

But then from the c) condition (y_n) is equivalent to the unit vector basis of ℓ_2 and from the above condition a subsequence is equivalent to the unit vector basis of ℓ_p, a contradiction.

The equivalence with d) requires further work and will have to wait. Note that b) $\Rightarrow$ d) since if (x_i) is a normalized weakly null sequence and we define $(x_\alpha)_{\alpha\in T_\infty}$ by $x_{(n_1,\dots,n_k)} = x_{n_k}$ then the branches of $(x_\alpha)_{\alpha\in T_\infty}$ coincide with the subsequences of (x_n). Note that the condition d) just says we may take the weight "w" in [JMST] to be "$\lim_i \|x_i\|_2$".

Theorem 7.12. *Let $X \subseteq L_p$, $2 < p < \infty$. If X does not embed into $\ell_p \oplus \ell_2$ then $(\sum \ell_2)_p \hookrightarrow X$.*

The idea of the proof is to produce a sequence of "skinny" ℓ_2's inside X, provided $X \not\hookrightarrow \ell_p \oplus \ell_2$.

Lemma. *Assume for some K and all n there exists $(x_i^n)_{n=1}^\infty \subseteq S_X$ with $\lim_i \|x_i^n\|_2 = \varepsilon_n \downarrow 0$ and $(x_i^n)_i$ is K-equivalent to the unit vector basis of ℓ_2. Then $(\sum \ell_2)_p \hookrightarrow X$.*

Sketch of proof. Note that if $y = \sum_i a_i x_i^n$ has norm 1 then, assuming as we may that $(x_i^n)_i$ is a block basis of (h_i) and $\|x_i^n\|_2 \approx \varepsilon_n$ then

$$\|y\|_2 \approx \Big(\sum a_i^2 \|x_i^n\|_2^2\Big)^{1/2} \lesssim K\varepsilon_n .$$

So we have a sequence of skinny $K - \ell_2$'s inside of X. We would like to have if $y^n \in [(x_i^n)_i]$ then they are essentially disjointly supported so $\|\sum y^n\| \sim (\sum \|y^n\|^p)^{1/p}$, as in the [KP] argument.

To achieve this we need a definition and a sublemma.

Definition. $A \subseteq L_p$ is *p-uniformly integrable* if $\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, m(E) < \delta\ \forall\, z \in A$, we have $\int_E |z|^p < \varepsilon$.

Sublemma. *Let $Y \subseteq L_p$, $2 < p < \infty$, with $Y \sim \ell_2$. There exists $Z \subseteq Y$ with S_Z p-uniformly integrable.*

This is proved in two steps. First showing a normalized martingale difference sequence (x_n) with $\{(x_n)\}$ p-uniformly integrable has $A = \{\sum a_i x_i : \sum a_i^2 \leq 1\}$ also p-uniformly integrable by a stopping time argument.

The general case is to use the subsequence splitting lemma to write a subsequence of an ℓ_2 basis as $x_i = y_i + z_i$ where the (y_i) are a p-uniformly integrable (perturbation of) a martingale difference sequence and the z_i's are disjointly supported and then use an averaging argument to get a block basis where the z_i's disappear. $\square$

Now we return to condition d) in Theorem 7.10 and recall by [JMST] every weakly null sequence in S_X has a subsequence (x_i) with for some $w \in [0,1]$,

$$\Big\| \sum a_i x_i \Big\| \overset{D_p}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee w\Big(\sum |a_i|^2 \Big)^{1/2}$$

and d) asserts that for some absolute C, $w \overset{C}{\sim} \lim_i \|x_i\|_2$. Now clearly we can assume that $w \geq \lim_i \|x_i\|_2$ and if d) fails we can use this to construct our ℓ_2's satisfying the lemma and thus obtain $(\sum \ell_2)_p \hookrightarrow X$.

It remains to show d) $\Rightarrow$ b) in Theorem 7.10 and this will complete the proof of Theorem 7.12. The idea is to use Burkholder's inequality using d) on nodes of a weakly null tree, following the scheme of [JMST] to accomplish this. That argument will obtain a branch $(x_n) = (x_{\alpha_n})$, $\alpha_n = (m_1, \dots, m_n)$ with

$$\Big\| \sum a_i x_i \Big\| \sim \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum w_i^2 a_i^2 \Big)^{1/2}$$

where $w_i \overset{C(p)}{\sim} \lim_n \|x_{(\alpha_n, n)}\|_2$ using d). $\square$

So we have the dichotomy for $X \subseteq L_p$, $2 < p < \infty$. Either

- $X \hookrightarrow \ell_p \oplus \ell_2$ or
- $(\sum \ell_2)_p \hookrightarrow X$.

In the latter case using L_p is stable we can get for all $\varepsilon > 0$, $(\sum \ell_2)_p \overset{1+\varepsilon}{\hookrightarrow} X$. In fact we can get $(\sum \ell_2)_p$ complemented in X via

Proposition 7.13. *For all n let $(y_i^n)_i$ be a normalized basic sequence in L_p, $2 < p < \infty$, which is K-equivalent to the unit vector basis of ℓ_2 and so that for $y_n \in [(y_i^n)_i]$,*

$$\Big\| \sum y_n \Big\| \overset{K}{\sim} \Big(\sum \|y_n\|^p \Big)^{1/p} .$$

Then there exists subsequences $(x_i^n)_i \subseteq (y_i^n)_i$, for each n, so that $[\{x_i^n : n, i \in \mathbb{N}\}]$ is complemented in L_p.

Proof. By [PR] each $[(y_i^n)_i]$ is $C(K)$-complemented in L_p via projections $P_n = \sum_m y_m^{n\,*}(x) y_m^n$. Passing to a subsequence and using a diagonal argument and perturbing we may assume there exists a blocking (H_m^n) of (h_i), in some order over all n, m, so that for all n, m, $\mathrm{supp}(y_m^{n\,*})$, $\mathrm{supp}(y_m^n) \subseteq H_m^n$. This uses $y_m^n \xrightarrow{w} 0$ and $y_m^{n\,*} \xrightarrow{w} 0$ (in $L_{p'}$) as $m \to \infty$ for each n. Set $Py = \sum_{n,m} y_m^{n\,*}(y) y_m^n$. We show P is bounded, hence a projection onto a copy of $(\sum \ell_2)_p$.

Let $y = \sum_{n,m} y(n,m)$, $y(n,m) \in H_n^m$.

$$\|Py\| = \sum_n \sum_m y_m^{n\,*}(y(n,m)) y_m^n \sim \Big(\sum_n \Big(\sum_m |y_m^{n\,*}(y(n,m))|^2 \Big)^{p/2} \Big)^{1/p} .$$

Now
$$\Big(\sum_m |y_m^{n\,*}(y(n,m))|^2\Big)^{1/2} \sim \|P_n y(n)\| \le C(K)\|y(n)\|$$
where $y(n) = \sum_m y(n,m)$. So
$$\|Py\| \le \bar C(K)\Big(\sum \|y_n\|^p\Big)^{1/p} \le \bar{\bar C}(K)\|y\| \ .$$

□

Remarks. The proof of Proposition 7.13 above is due to Schechtman. He also proved by a different much more complicated argument that the proposition extends to $1 < p < 2$.

In [HOS] the proofs of all the results are also considered using Aldous' [Ald] theory of random measures. We are able to show if $(\sum \ell_2)_p \hookrightarrow X \subseteq L_p$, $2 < p < \infty$, then given $\varepsilon > 0$ there exists $(\sum Y_n)_p \overset{1+\varepsilon}{\hookrightarrow} X$, $d(Y_n,\ell_2) < 1+\varepsilon$ and moreover: there exist disjoint sets $A_n \subseteq [0,1]$ with for all n, $y \in Y_n$, $\|y|_{A_n}\| \ge (1-\varepsilon 2^{-n})\|y\|$ and $[Y_n : n \in \mathbb{N}]$ is $(1+\varepsilon)$ C_p^{-1} complemented in L_p where C_p is the norm of a symmetric normalized Gaussian random variable in L_p. This is best possible by [GLR].

We can also deduce the [JO3] result: $X \subseteq L_p$, $2 < p < \infty$, and X is a quotient of a subspace of $\ell_p \oplus \ell_2 \Rightarrow X \hookrightarrow \ell_p \oplus \ell_2$, by showing that such an X cannot contain $(\sum \ell_2)_p$.

The [KP], [JO2] results yield for $X \subseteq L_p$, $2 < p < \infty$

- X is asymptotic $\ell_p \Rightarrow X \hookrightarrow \ell_p$
- X is asymptotic $\ell_2 \Rightarrow X \hookrightarrow \ell_2$.

Definition. X is *asymptotically* $\ell_p \oplus \ell_2$ if $\exists\ K\ \forall\ n \forall\ (e_i)_1^n \in \{X\}_n\ \exists\ (w_i)_1^n$ with
$$\Big\|\sum_1^n a_i e_i\Big\| \overset{K}{\sim} \Big(\sum_1^n |a_i|^p\Big)^{1/p} \vee \Big(\sum_1^n w_i^2 a_i^2\Big)^{1/2} \ .$$

Proposition 7.14. *Let $X \subseteq L_p$, $2 < p < \infty$. X is asymptotically $\ell_p \oplus \ell_2$ iff $X \hookrightarrow \ell_p \oplus \ell_2$.*

This follows easily from our results by showing that $(\sum \ell_2)_p$ is not asymptotically $\ell_p \oplus \ell_2$.

Problem. Let $X \subseteq L_p$, $p > 2$. Give an intrinsic characterization of when $X \hookrightarrow (\sum \ell_2)_p$.

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